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Periodic Points for Expanding Maps and for Their Extensions

TESI DI PERFEZIONAMENTO

SCUOLA NORMALE SUPERIORE

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by

Roberto Tauraso

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Introduction

Ayudadme a comprender lo que os digo,
y os lo explicaré más despacio.

(Antonio Machado, *Juan de Mairena*)

The aim of this thesis is to give a complete exposition of the main topological and ergodic properties of the expanding maps and provide some new results about their periodic points sets. I briefly state now the main results and refer the reader to the introductions of the various Chapters for more detailed comments.

It is a common practice in topological dynamics to examine the results of differentiable dynamics and try to establish them in a not necessarly differentiable setting. This is the case of the expanding maps, the simplest example of non-invertible hyperbolic maps. These maps were introduced in 1969 by M. Shub, and subsequently D. Ruelle found a new definition without the differentiability hypothesis which is sufficient to recover their dynamical properties.

In Chapter 1 a particular attention has been given to the various equivalent definitions of expanding maps. First we collect their first topological properties when the domain of the maps is a compact connected metric space. Any expanding map F is a self-covering with degree $N \geq 2$ (when the domain X contains at least two points). Moreover, the set of fixed points turns out to be always non-empty and finite, and, the set of periodic points, $\mathrm{Per}_X(F)$, is dense in X. Then we investigate the particular case in which the domain X is a closed topological manifolds $\mathcal{M}$. With this assumption, these maps have a *chaotic* behaviour which however can be fruitfully studied exploiting the *Conjugation Theorem*, proved by K. Hiraide in 1988. This result says that a closed manifold which admits some expanding map is necessarly homeomorphic to an infra-nil-manifold $\mathcal{M}'$, i. e. the quotient space N/Γ, where N is a connected, simply connected nilpotent Lie group and Γ is a cocompact, torsion free, discrete subgroup of $\mathsf{K}\rtimes\mathsf{N}$ with K a compact subgroup of the group of automorphisms of N. Moreover, any expanding map F of $\mathcal{M}$ can be topologically conjugated to an expanding endomorphism Φ of $\mathcal{M}'$, that is, a map induced by an automorphism φ of $\mathsf{K}\rtimes\mathsf{N}$ such that $\varphi(\Gamma) \subset \Gamma$. Chapter 1 ends with a comparison between an expanding map of degree N and one-sided shift on a set of N elements.

Chapter 2 is devoted to the description of a scheme, due essentially to P. Walters and here adapted to the case of the expanding maps, which provides normalized invariant measures for an expanding map F. The map F turns out to be *exact* with respect to these measures, that is, every set belonging to the intersection $\bigcap_{k=0}^{\infty} F^{-k}(\mathrm{Bor}(\mathcal{M}))$, where $\mathrm{Bor}(\mathcal{M})$ is the set of all borelians of $\mathcal{M}$, has measure equal to either 0 or 1 (exactness is stronger then ergodicity). By this scheme, based on the transfer operator, we find two remarkable measures which will be needed subsequently: the invariant measure μ_0 with maximal entropy and, in the differentiable case, the invariant measure μ_F that is equivalent to the Lebesgue measure.

In Chapter 3, after a brief exposition of the Nielsen index theory, the set of periods $\mathcal{P}(F)$ of an expanding map F is investigated; $\mathcal{P}(F)$ is the set of all natural numbers k, for which, there exists at least a periodic point of the map of period exactly k. The first result whereby $\mathcal{P}(F)$ is cofinite, i. e. the set $\mathbb{N}^*/\mathcal{P}(F)$ is finite, is established using the invariant measure with maximal entropy. Thereafter, a closer look is given to the case of flat manifolds. All elements of this subclass of infra-nil-manifolds (in which case $\mathsf{N} = \mathbb{R}^n$), are shown to admit always an expanding map. Using the Conjugation Theorem, we prove that on a flat manifold, the sets of periods of its expanding maps are uniformly cofinite. Then, we complete the study of the structure of the set of periods, started by J. Llibre for the torus up to dimension 3, by showing that for a map F expanding on an infra-torus up to dimension 3, $\mathcal{P}(F)$ is always equal to $\mathbb{N}^*$. A brief discussion on the set of periods of an expanding map on a nil-manifold is also given.

Chapter 4, faces the problem of finding common fixed points for two commuting expanding maps on a nil-manifold. By finding a common homeomorphism which conjugates the two maps to two expanding endomorphisms, an answer is given when the cardinalities of their fixed points sets are relatively prime. This condition yields also the uniqueness of the common fixed point. Using these facts, when the domain is a circle, we recover a known result about the regularity of this homeomorphism.

Chapter 5 is entirely devoted to the differentiable case and to the extensions of expanding maps from the torus to the polydisc. In the one-dimensional case, we consider the open disc Δ bounded by the circle $\mathbf{T}^1$, and, we show that all C^1-extensions to $\overline{\Delta}$ of any expanding map on $\mathbf{T}^1$ have always a fixed point in Δ if this map preserves the orientation, whereas this is not true in general when the orientation is reversed. Thereafter, given the torus $\mathbf{T}^n$ of (real) dimension n, we consider the open n-disc Δ^n of (complex) dimension n of which $\mathbf{T}^n$ is the Shilov boundary and we investigate the holomorphic extension to $\overline{\Delta}^n$ of an expanding map on $\mathbf{T}^n$, showing that this extension has a unique fixed point inside Δ^n. This Chapter ends with some results concerning the total number of fixed points on the topological boundary of Δ^n.

I would like to close this introduction with sincere thanks to my advisor Edoardo Vesentini, who followed the slow growth of this work, providing useful suggestions and continuous encouragement. I am indebted for corrections and various comments to Marco Abate and Alberto Abbondandolo, who, in different moments, read drafts of this thesis and helped me to improve it.

My gratitude goes to the Scuola Normale Superiore, that supported me during my studies and gave me the opportunity to live in a very stimulating environment (not only for mathematics). Finally, I thank Graziano Gentili and the staff of the S.I.S.S.A. of Trieste for the kind hospitality and the generous grant, which allowed me to complete the writing of this thesis.

Pisa, April 1996 Roberto Tauraso

1

Topological properties

The expanding maps have been first introduced in a differentiable setting by M. Shub in [Sh1]. Later on, D. Ruelle in [Rue], proposed the following more general definition. Let X be a topological space. An open continuous map $F : X \to X$ is *expanding* if there exist a metric d compatible with the topology of X and costants $\epsilon_0 > 0$, $\lambda > 1$ such that for $x, x' \in X$

$$d(x, x') \leq \epsilon_0 \quad \text{implies} \quad d(F(x), F(x')) \geq \lambda d(x, x').$$

The number λ is the *expanding constant* of F and the set of the expanding maps on X is denoted by $\mathcal{E}(X)$. In this Chapter, we provide a complete survey of the topological properties of this class of dynamical systems.

First we assume that the set X is a connected compact metrizable space and we give two equivalent definitions of expanding map. In particular, one of them is independent of the compatible metric on X. Then we prove that any $F \in \mathcal{E}(X)$ is a self-covering with degree $N \geq 2$ when X contains at least two points. Thanks to the pseudo orbit tracing property we are able to give a first description of the set of periodic points $\mathrm{Per}_X(F)$ of a map $F \in \mathcal{E}(X)$. The set of fixed points turns out to be always non-empty and finite, and, therefore $\mathrm{Per}_X(F)$ is countable. Moreover, $\mathrm{Per}_X(F)$ is dense in X.

Then we take the space X as a *closed* manifold $\mathcal{M}$, i. e. an n-dimensional connected compact topological manifold without boundary (we will prove that if $\mathcal{M}$ has non-empty boundary then it does not admit any expanding map). If $F \in \mathcal{E}(\mathcal{M})$, we will manage to find a compatible metric $\tilde{d}$ on the universal covering $\widetilde{\mathcal{M}}$ with respect to which a lifting $\widetilde{F}$ of F is an expanding homeomorphism of $\widetilde{\mathcal{M}}$ and the elements of $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ are isometries. These facts imply that the set of points of $\mathcal{M}$, which have dense orbits for F, is dense whereas this is not true, in general, when X is only a metric space. Thus expanding maps are *chaotic* according to the definition of Devaney (see for example [BBCDS]).

The first step towards the classification of expanding maps is that each $F \in \mathcal{E}(\mathcal{M})$ is characterized by the homomorphism $\widetilde{F}^\sharp$ induced by F on $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$. More precisely, two expanding maps $F \in \mathcal{E}(\mathcal{M})$, $\Phi \in \mathcal{E}(\mathcal{M}')$ are topologically conjugate, i. e. $\alpha_0 \circ F = \Phi \circ \alpha_0$ for a homeomorphism $\alpha_0 : \mathcal{M} \to \mathcal{M}'$, if there is an isomorphism $\psi : \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M}) \to \mathcal{D}(\widetilde{\mathcal{M}}'/\mathcal{M}')$ such that $\psi \circ \widetilde{F}^\sharp = \widetilde{\Phi}^\sharp \circ \psi$. In particular, two homotopic expanding maps are topologically conjugate.

Moreover, we prove, the following preparatory results: if a closed manifold $\mathcal{M}$ admits an expanding map, then, $\mathcal{M}$ is *aspherical*, i. e. the homotopy groups $\pi_i(\mathcal{M}) = 0$ for $i \geq 2$, and, the group $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ is torsion free and has polynomial growth. Then, we introduce for a smooth closed manifold $\mathcal{M}$ the subclass of expanding maps which correspond to the original definition of M. Shub. A C^1-map $F : \mathcal{M} \to \mathcal{M}$ is called *differentiably expanding* if there are a compatible riemannian metric $||\cdot||$ on $T\mathcal{M}$ and a constant $\lambda > 1$ such that at any point $x \in \mathcal{M}$

$$||(D_x F)v|| \geq \lambda ||v|| \qquad \forall v \in T_x\mathcal{M}.$$

The set of the differentiably expanding maps on $\mathcal{M}$ is denoted by $\mathcal{E}^1(\mathcal{M})$. Afterwards, we draw our attention to a special class of riemannian closed manifold, called *infra-nil-manifolds*. If K is a compact subgroup of the group of automorphisms of a connected, simply connected nilpotent Lie group N and Γ is a cocompact, torsion free, discrete subgroup of $\mathsf{K} \ltimes \mathsf{N}$ then the quotient space $\mathcal{M}' = \mathsf{N}/\Gamma$ is, by definition, an infra-nil-manifold; the finite group $\Psi = \Gamma/(\Gamma \cap (\{\mathbb{I}\} \ltimes \mathsf{N}))$ is the *holonomy group* of the manifold. The universal covering $\widetilde{\mathcal{M}}'$ of $\mathcal{M}'$ is N and $\Gamma = \mathcal{D}(\widetilde{\mathcal{M}}'/\mathcal{M}') \simeq \pi_1(\mathcal{M}')$. A map $\Phi : \mathcal{M}' \to \mathcal{M}'$ is called an *endomorphism* of $\mathcal{M}'$ if it is induced by an automorphism φ of $\mathsf{K} \ltimes \mathsf{N}$ such that $\varphi(\Gamma) \subset \Gamma$. Moreover it belongs to $\mathcal{E}^1(\mathcal{M}')$ iff the differential of $\widetilde{\Phi}$ at the identity element of N has all the eigenvalues greater than one in absolute value.

Then we arrive to the *Conjugation Theorem* proved by K. Hiraide in [Hira1], which says that if $F \in \mathcal{E}(\mathcal{M})$, there exist an infra-nil-manifold $\mathcal{M}'$ and an endomorphism $\Phi \in \mathcal{E}^1(\mathcal{M}')$ such that F and Φ are topologically conjugate. The Chapter ends with a comparison between an expanding map of degree N and one-sided shift on a set of N elements.

§1.1 Self-covering maps.

Let X be a topological space. A continuous surjection $F : X \to X$ is called a *self-covering map* if, for $y \in X$, there exists an open neighborhood V_y of y such that

$$F^{-1}(V_y) = \bigcup_{i \in \mathcal{I}} U_i$$

where $\{U_i\}_{i \in \mathcal{I}}$ is a family of mutually disjoint open sets, and $F_{|U_i} : U_i \to V_y$ is a homeomorphism. A continuous map $F : X \to X$ is a *local homeomorphism* if, for $x \in X$, there is an open neighborhood U_x of x such that $F(U_x)$ is open and $F_{|U_x} : U_x \to F(U_x)$ is a homeomorphism. It is clear that every covering map is a local homeomorphism.

The following theorem shows that adding some hypotheses on the space X, also the converse is true.

Theorem 1.1.1 [Ei] *Let (X, d) be a compact connected metric space and $F : X \to X$ be a continuous map. If F is a local homeomorphism, then F is a N-to-1 map with $N \geq 1$ and there exist two positive costants η_0 and ρ_0 such that, if $D \subset X$ with $\operatorname{diam}(D) \leq \eta_0$ then:*

(a) $F^{-1}(D) = \bigcup_{i=1}^{N} D_i$;

(b) F maps each D_i homeomorphically onto D;

(c) if $i \neq i'$ then $d(D_i, D_{i'}) \geq \rho_0$;

(d) for all $\epsilon > 0$ there exists a positive number $\eta(\epsilon) \leq \eta_0$ such that

$$\operatorname{diam}(D) \leq \eta(\epsilon) \;\; \textit{implies} \;\; \operatorname{diam}(D_i) \leq \epsilon \;\; \textit{for} \;\; i = 1 \dots N.$$

Hence, F is a self-covering map.

Proof. Since F is continuous and X is a compact metric space, $F(X)$ is closed in X. Moreover $F(X)$ is also open in X because F is a local homeomorphism; since X is connected then $F(X) = X$, i. e. F is surjective.

Now, we first prove that there exists $\rho_0 > 0$ such that

$$\text{if } x \neq y \text{ and } F(x) = F(y) \text{ then } d(x, y) \geq 2\rho_0. \tag{1}$$

If this is not true, then for all $k \geq 1$ there exist x_k, y_k in X such that $x_k \neq y_k$, $F(x_k) = F(y_k)$ and $d(x_k, y_k) \leq \frac{1}{k}$. Since X is compact we can assume that $x_k \to x$ and $y_k \to y$ as $k \to \infty$, then $x = y$. Since F is a local homeomorphism, there exists an open neighborhood U_x of x such that $F(U_x)$is open and $F_{|U_x} : U_x \to F(U_x)$ is a homeomorphism. This means that, for sufficiently large k, $x_k, y_k \in U_x$ and $F(x_k) = F(y_k)$. This is a contradiction.

Let $j \geq 1$ and define the set

$$X_j \stackrel{\text{def}}{=} \{y \in X : \text{card}(F^{-1}(y)) = j\}.$$

Then $X_j \cap X_{j'} = \emptyset$ whenever $j \neq j'$. Moreover, by (1), since X is compact and F is surjective, $X = \bigcup_{j \geq 1} X_j$.

Next, we prove that each X_j is open. Thus, since X is connected, there exists an integer $N \geq 1$ such that $X = X_N$. If $y \in X_j$, then $F^{-1}(y) = \{x_1, \cdots, x_j\}$. We can choose open neighborhoods U_{x_i} of x_i for $i = 1, \dots, j$ such that $F_{|U_{x_i}}$ is a homeomorphism and $U_{x_i} \cap U_{x_{i'}} = \emptyset$ if $i \neq i'$. It is clear that

$$y \in \bigcap_{i=1}^{j} F(U_{x_i}) \subset \bigcup_{i \geq j} X_i.$$

We claim that $\bigcap_{i=1}^{j} F(U_{x_i})$ contains a neighborhood V of y such that $y \in X_j$. In fact, if this is false, for all $k \geq 1$ there is $y_k \in \bigcap_{i=1}^{j} F(U_{x_i})$ for which

$$d(y, y_k) \leq \frac{1}{k} \quad \text{and} \quad y_k \in \bigcup_{i \geq j+1} X_i.$$

Then $F^{-1}(y_k) = \{x_{k,1}, \cdots, x_{k,j}, x_{k,j+1}, \cdots\}$ with $x_{k,i} \in U_{x_i}$ and $x_{k,i} \to x_i$ as $n \to \infty$ for $i = 1, \cdots, j$.

Since X is compact we can assume that also the sequence $\{x_{k,j+1}\}_{k \geq 1}$ converges, say to x_{j+1}. Moreover x_{j+1} is different from x_i for each $i = 1, \cdots, j$, because, by (1), $d(x_{k,j+1}, x_{k,i}) \geq 2\rho_0$ for all $k \geq 1$. But $F(x_i) = y$ for each $i = 1, \cdots, j+1$, hence $y \in \bigcup_{i \geq j+1} X_i$. This is a contradiction.

Since X is compact and F is a local homeomorphism, we can choose a finite open cover $\mathcal{U} = \{U_1, \cdots, U_M\}$ satisfying, for $m = 1, \cdots, M$:

i) $F(U_m)$ is open,

ii) $F : \overline{U}_m \to F(\overline{U}_m)$ is injective,

iii) $\operatorname{diam}(\overline{U}_m) \leq \frac{\rho_0}{2}$.

Let $y \in X$ and define

$$V(y) \stackrel{\text{def}}{=} \cap\{F(U_m) : y \in F(U_m) \text{ for some } U_m \in \mathcal{U}\};$$

then $V(y)$ is an open neighborhood of y. Since X is compact, we can find another finite open cover $\mathcal{V} = \{V(y_1), \cdots, V(y_l)\}$.

(a) Let $\eta_0 > 0$ be a Lebesgue number of the open cover $\mathcal{V}$ and let $D \subset X$ be such that $\operatorname{diam}(D) \leq \eta_0$. Thus $D \subset V(y)$ for some $y \in \{y_1, \cdots, y_l\}$ and $F^{-1}(y) = \{x_1, \cdots, x_N\}$ with each $x_i \in U_{m_i}$ for some $U_{m_i} \in \mathcal{U}$.

Notice that, by ii), $F^{-1}(y) \cap U_{m_i} = x_i$. We now define $D_i = U_{m_i} \cap F^{-1}(D)$ for $i = 1, \cdots, N$; thus

$$F(D_i) = F(U_{m_i} \cap F^{-1}(D)) \subset F(U_{m_i}) \cap D = D. \tag{2}$$

Moreover, if $z \in D$, there exists some $x \in U_{m_i}$ such that $z = F(x)$ and

$$z = F(x) \in F(U_{m_i} \cap F^{-1}(z)) \subset F(U_{m_i}) \cap F^{-1}(D)) = F(D_i).$$

Together with (2), we have proved that $F(D_i) = D$ for $i = 1, \cdots, N$.

(b) Since for $i = 1, \cdots, N$, $D_i \subset U_{m_i}$ and $D = F(D_i) \subset F(U_{m_i})$, by ii) we obtain that F maps D_i homeomorphically onto D.

(c) If $i \neq i'$, then, by iii) and (1),

$$d(D_i, D_{i'}) \geq d(x_i, x_{i'}) - \mathrm{diam}(U_{m_i}) - \mathrm{diam}(U_{m_{i'}}) \geq 2\rho_0 - \frac{\rho_0}{2} - \frac{\rho_0}{2} = \rho_0.$$

(d) Let $\epsilon > 0$. By ii) we can define, for $m = 1, \ldots, M$, the inverse functions of $F_{|\overline{U}_m}$, $G_m : F(\overline{U}_m) \to \overline{U}_m$. Since each G_m is continuous on the compact set $F(\overline{U}_m)$, it is also uniformly continuous: there exists $\delta_m > 0$ such that

$$d(x, x') \leq \delta_m \ \text{ implies } \ d(G_m(x), G_m(x')) \leq \epsilon.$$

Let $\eta(\epsilon) \stackrel{\text{def}}{=} \min\{\delta_1, \ldots, \delta_M, \eta_0\}$ and take $D \subset X$ such that $\mathrm{diam}(D) \leq \eta(\epsilon)$. Then, by (a), $D_i = U_{m_i} \cap F^{-1}(D)$. For $y, y' \in D_i$ then $F(y), F(y') \in F(U_{m_i}) \cap D$ and $d(F(y), F(y')) \leq \eta(\epsilon)$. Therefore

$$d(y, y') = d(G_m(F(y)), G_m(F(y'))) \leq \epsilon.$$

Q.E.D.

§1.2 Expanding maps: definition and first properties.

Let X be a topological space.

Definition 1.2.1 An open continuous map $F : X \to X$ is *expanding* if there exist a metric d compatible with the topology of X and constants $\epsilon_0 > 0$, $\lambda > 1$ such that for $x, x' \in X$

$$d(x, x') \leq \epsilon_0 \quad \text{implies} \quad d(F(x), F(x')) \geq \lambda d(x, x'). \tag{3}$$

The number λ is the *expanding constant* of F and we will denote by $\mathcal{E}(X)$ the set of all maps expanding on X.

Remark 1.2.2 Note that the definition, due to D. Ruelle (see [Rue]), is invariant by topological conjugation, that is, if $F \in \mathcal{E}(X)$ and $h : X \to Y$

is a homeomorphism, then the map $G \stackrel{\text{def}}{=} h \circ F \circ h^{-1}$ is expanding on the topological space Y.

In fact, $d'(\cdot,\cdot) \stackrel{\text{def}}{=} d(h^{-1}(\cdot), h^{-1}(\cdot))$ is a metric on Y and if $y, y' \in Y$ with $d'(y,y') = d(h^{-1}(y), h^{-1}(y')) \leq \epsilon_0$ then

$$d'(G(y), G(y')) = d(F(h^{-1}(y)), F(h^{-1}(y'))) \geq \lambda d(h^{-1}(y), h^{-1}(y')) = \lambda d'(y,y').$$

Therefore $G \in \mathcal{E}(Y)$ with the same expanding constant of F.

From now on we assume that X is a connected, compact and metrizable.

Proposition 1.2.3 [CoRe] *Let $F : X \to X$ be an open continuous map and let $k \geq 1$. Then $F \in \mathcal{E}(X)$ iff $F^k \in \mathcal{E}(X)$.*

Proof. Assume that $F \in \mathcal{E}(X)$. Since F is uniformly continuous on X, there exists a $\delta > 0$ such that for $i = 1, \ldots, k$

$$d(x,x') \leq \delta \quad \text{implies} \quad d(F^i(x), F^i(x')) \leq \epsilon_0.$$

Let $\epsilon_0' \stackrel{\text{def}}{=} \min\{\delta, \epsilon_0\} > 0$ and $\lambda' \stackrel{\text{def}}{=} \lambda^k > 1$. Then applying k times (3) we find that $F^k \in \mathcal{E}(X)$ with costants ϵ_0', λ' and the same metric d.

Now, assume that $F^k \in \mathcal{E}(X)$. Let $\lambda' \stackrel{\text{def}}{=} \lambda^{\frac{1}{k}}$ and define the compatible metric on X

$$d'(x,x') \stackrel{\text{def}}{=} \sum_{i=0}^{k-1} \frac{1}{\lambda'^i} d(F^i(x), F^i(x')) \qquad \forall x, x' \in X.$$

Then $F \in \mathcal{E}(X)$ with costants ϵ_0, λ' and the metric d'. Q.E.D.

The following theorem gives two equivalent definitions of the expanding property. We will need the *Frink's metrization lemma*, whose proof can be found in [Ke] p. 185.

Lemma 1.2.4 [Ke] *Let X be a compact space. If there is a nested sequence $\{U_k\}_{k\geq 0}$ of open symmetric neighborhoods of the diagonal set $\Delta_X \stackrel{\text{def}}{=} \{(x,x) : x \in X\}$ of $X \times X$ such that:*

i) $U_0 = X \times X$, $\cap_{k\geq 0} U_k = \Delta_X$;

ii)[1] $U_{k+1} \diamond U_{k+1} \diamond U_{k+1} \subset U_k$ *for* $k \geq 1$.

Then there is a compatible metric d *for* X *such that, for* $k \geq 1$

$$U_k \subset \{(x, x') \ : \ d(x, x') < \frac{1}{2^k}\} \subset U_{k-1}.$$

Theorem 1.2.5 [Re] *If* $F : X \to X$ *is an open continuous map, the following statements are equivalent:*

(a) F *is expanding;*

(b) F *is* locally expansive*: there exist a compatible metric* d' *for* X *and a constant* $\epsilon'_0 > 0$ *such that, for* $x, x' \in X$,

$$0 < d'(x, x') \leq \epsilon'_0 \quad \textit{implies} \quad d'(F(x), F(x')) > d'(x, x');$$

(c) F *is* positively expansive*: there exist a compatible metric* d'' *for* X *and a constant* $c > 0$ *such that*

$$x \neq x' \ \ \textit{implies} \ \exists k \geq 0 \ \ \textit{such that} \ \ d''(F^k(x), F^k(x')) \geq c.$$

Proof. It is obvious that (a) implies (b) with $d' \stackrel{\text{def}}{=} d$ and $\epsilon'_0 \stackrel{\text{def}}{=} \epsilon_0$. Now, we prove that (b) implies (c) with $d'' \stackrel{\text{def}}{=} d'$ and $c \stackrel{\text{def}}{=} \epsilon'_0$.

Let $x \neq x'$ and assume that

$$d'(F^k(x), F^k(x')) < \epsilon'_0 \qquad \forall k \geq 0.$$

Define the map $h = F \times F$, and let $S \stackrel{\text{def}}{=} \{h^k(x, x') \ : \ k \geq 0\} \subset X \times X$. Let (x_0, x'_0) be a point in the compact set $\overline{S}$ at which the continuous map d attains its maximum. Then

$$\epsilon'_0 \geq d'(x_0, x'_0) \geq d'(x, x') > 0.$$

Since F is locally expansive and $h(\overline{S}) \subset \overline{S}$ we have

$$d'(x_0, x'_0) < d'(F(x_0), F(x'_0)) \leq d'(x_0, x'_0),$$

[1] If U, V are two subsets of $X \times X$ then $(x, x') \in U \diamond V$ iff there exists $y \in X$ such that $(x, y) \in U$ and $(y, x') \in V$. It is easy to verify that the operation $\diamond$ is associative.

hence a contradiction.

Now, suppose that F is positively expansive. Let $V_0 \stackrel{\text{def}}{=} X \times X$ and

$$V_k \stackrel{\text{def}}{=} \{(x, x') \ : \ d''(F^i(x), F^i(x')) < c, \ 0 \le i < k\}.$$

The sequence $\{V_k\}_{k \ge 0}$ is a nested sequence of symmetric neighborhoods of the diagonal set $\Delta_X = \{(x, x) \ : \ x \in X\}$. Since F is positively expansive, $\cap_{k=0}^{\infty} V_k = \Delta_X$.

Let $\delta > 0$ be such that $B''_{X \times X}(\Delta_X, \delta) \subset V_1$. Since X is compact, there exists k_0 such that

$$V_{1+k_0} \subset B''_{X \times X}(\Delta_X, \frac{1}{3}\delta).$$

Let $U_0 \stackrel{\text{def}}{=} X \times X$ and for $k \ge 1$, $U_k \stackrel{\text{def}}{=} V_{1+(k-1)k_0}$. The sequence $\{U_k\}_{k \ge 0}$ satisfies the hypotheses of the Frink's metrization lemma. In fact, it suffices to prove that the relation ii) holds. It holds by costruction for $k = 0, 1$.

Let $(x, x') \in U_{k+1} \diamond U_{k+1} \diamond U_{k+1}$ for $k \ge 2$. Then there exist points y and z such that (x, y), (y, z) and (z, x') belong to U_{k+1}.

Let $h \stackrel{\text{def}}{=} F \times F$, then

$$h(V_{k+1}) = V_k \cap h(V_1) \qquad \forall k \ge 0.$$

Thus, for $0 \le i \le (k-1)k_0$,

$$h^i((x, x')) \in h^i(U_{k+1}) \diamond h^i(U_{k+1}) \diamond h^i(U_{k+1}) \subset V_{1+kk_0-i} \diamond V_{1+kk_0-i} \diamond V_{1+kk_0-i}.$$

Since $V_{1+kk_0-i} \subset V_{1+k_0} = U_2$, then

$$h^i((x, x')) \in U_2 \diamond U_2 \diamond U_2 \subset U_1.$$

Therefore, by the definition of U_1, $d''(F^i(x), F^i(x')) < c$ for $0 \le i \le (k-1)k_0$ and thus $(x, x') \in V_{1+(k-1)k_0} = U_k$.

By the Frink's metrization lemma, there exists a compatible metric d for X such that

$$U_k \subset \{(x, x') \ : \ d(x, x') < \frac{1}{2^k}\} \subset U_{k-1} \ \text{ for } k \ge 1. \tag{4}$$

Suppose that $0 < d(x,x') < \frac{1}{32}$. Then, by the latter inclusions, there exists $k \geq 3$ such that $(x,x') \in U_{k+1} \setminus U_{k+2}$. Then

$$\frac{1}{2^{k+3}} \leq d(x,x') < \frac{1}{2^{k+1}}. \tag{5}$$

Moreover, since $(x,x') \in U_{k+1} \setminus U_{k+2} = V_{1+kk_0} - V_{1+(k+1)k_0}$, there exists i such that

$$kk_0 < i < (k+1)k_0 \text{ and } d''(F^i(x), F^i(x')) > c.$$

Let $(y,y') = (F^{3k_0}(x), F^{3k_0}(x'))$. Then

$$d''(F^{i-3k_0}(y), F^{i-3k_0}(y')) = d''(F^i(x), F^i(x')) > c,$$

and therefore $(y,y') \notin V_{1+(k-2)k_0} = U_{k-1}$ because $1 + (k-2)k_0 > i - 3k_0$. Hence, by (4) and (5),

$$d(F^{3k_0}(x), F^{3k_0}(x')) > \frac{1}{2^k} > 2d(x,x'),$$

that is $F^{3k_0} \in \mathcal{E}(X)$. By Proposition 1.2.3 also $F \in \mathcal{E}(X)$. Q.E.D.

Remark 1.2.6 Note that, since X is compact, the definition of positively expansive map is independent of the compatible metric for X. This fact can be very useful, if we want to check that an open continuous map is expanding.

Theorem 1.2.7 [AoHi] *Any $F \in \mathcal{E}(X)$ is a self-covering map with degree $N \geq 1$, and there exists $0 < \epsilon_1 \leq \epsilon_0$ such that, if $D \subset X$ with $\mathrm{diam}(D) \leq \epsilon_1$, then, for all $k \geq 1$,*

(a) $F^{-k}(D) = \bigcup_{i=1}^{N^k} D_i^{(k)}$;

(b) F^k *maps each* $D_i^{(k)}$ *homeomorphically onto* D;

(c) $\mathrm{diam}(D_i^{(k)}) \leq \frac{\epsilon_1}{\lambda^k}$ *for* $i = 1 \ldots N^k$.

Moreover, $N = 1$ iff X consists of one point.

Proof. Since F is open and, by (3), is locally injective, then it is a local homeomorphism. Therefore, by Theorem 1.1.1, F is a self-covering map with degree $N \geq 1$.

Moreover, by the same theorem there is a positive number $\eta_1 \leq \eta_0$ such that, if a set D has $\operatorname{diam}(D) \leq \eta_1$, then $\operatorname{diam}(D_i^{(1)}) \leq \epsilon_0$ for each component $D_i^{(1)}$ of $F^{-1}(D)$. Let $\epsilon_1 \stackrel{\text{def}}{=} \min\{\eta_1, \epsilon_0\}$ and assume that $\operatorname{diam}(D) \leq \epsilon_1$. By (3), if $y, y' \in D_i^{(1)}$, then

$$\epsilon_1 \geq d(F(y), F(y')) \geq \lambda d(y, y')$$

and therefore, $\operatorname{diam}(D_i^{(1)}) \leq \frac{\epsilon_1}{\lambda}$. Since $\lambda > 1$, we can use the same argument for each $D_i^{(1)}$ instead of D and continuing in this fashion we prove (a), (b) and (c).

Now, assume that F is a homeomorphism and define the sets

$$\mathcal{S}^+ \stackrel{\text{def}}{=} \{F^k \ : \ k \geq 0\} \ \text{ and } \ \mathcal{S}^- \stackrel{\text{def}}{=} \{F^{-k} \ : \ k \geq 0\}.$$

If $d(x, x') \leq \epsilon_1$ then letting $D \stackrel{\text{def}}{=} \{x, x'\}$, by the above arguments,

$$d(F^{-k}x, F^{-k}x')) \leq \frac{\epsilon_1}{\lambda^k}.$$

Therefore $\mathcal{S}^-$ is uniformly equi-continuous. Since X is compact, by Ascoli's theorem $\mathcal{S}^-$ is totally bounded in $C(X, X)$ with respect to the sup-metric $\hat{d}$.

Define the map $\varphi : \mathcal{S}^- \to \mathcal{S}^+$ by $\varphi(F^{-k}) = F^k$ for $k \geq 0$. Then φ is a $\hat{d}$-isometry: in fact, for $k, k' \geq 0$,

$$\hat{d}(F^k, F^{k'}) = \sup_{x \in X}\{d(F^k(x), F^{k'}(x))\} = \sup_{x \in F^{k+k'}(X)}\{d(F^{-k'}(x), F^{-k}(x))\} = \hat{d}(F^{-k}, F^{-k'}).$$

Therefore, also $\mathcal{S}^+$ is totally bounded and, again by Ascoli's theorem, $\mathcal{S}^+$ is uniformly equi-continuous. So, there is a $\delta > 0$ such that

$$d(x, x') \leq \delta \ \text{ implies } \ d(F^k(x), F^k(x')) \leq \epsilon_0 \qquad \forall k \geq 0,$$

and by (3) this means

$$\epsilon_0 \geq d(F^k x, F^k x') \leq \lambda^k d(x, x').$$

Since $\lambda > 1$, X consists of isolated points and X is reduced to one point because X is connected. Q.E.D.

Definition 1.2.8 Given $\delta > 0$, a δ-*pseudo orbit* of a map $F : X \to X$ is a sequence $\{x_k\}_{k\geq 0}$ such that $d(F(x_k), x_{k+1}) < \delta$ for $k \geq 0$.

The map F has the *pseudo orbit tracing property* (abbreviated P.O.T.P.) if, for any $\epsilon > 0$, there is $\delta > 0$ such that for every δ-pseudo orbit of F, $\{x_k\}_{k\geq 0}$ can be ϵ-*traced* by some point $x \in X$, i. e. $d(F^k(x), x_k) < \epsilon$ for $k \geq 0$.

Theorem 1.2.9 [AoHi] *Any $F \in \mathcal{E}(X)$ has the P.O.T.P..*

Proof. Assuming that $0 < \epsilon \leq \epsilon_0$, since F is uniformly continuous, we can choose δ such that

$$0 < \delta\frac{\lambda}{\lambda-1} < \epsilon \ \text{ and } \ B(F(x), \delta\frac{\lambda}{\lambda-1}) \subset F(B(x,\epsilon)) \quad \forall x \in X.$$

If $\{x_k\}_{k\geq 0}$ is a δ-pseudo orbit of F, fix $k \geq 1$ and let $x^{(0)} = x_k$.

Since $\lambda > 1$ then

$$x^{(0)} \in B(F(x_{k-1}), \delta) \subset B(F(x_{k-1}), \delta\frac{\lambda}{\lambda-1}) \subset F(B(x_{k-1}, \epsilon)),$$

and there exists $x^{(1)} \in B(x_{k-1}, \epsilon)$ such that $F(x^{(1)}) = x^{(0)}$.

Moreover, by (3),

$$\begin{aligned} d(x^{(1)}, F(x_{k-2})) &\leq d(x^{(1)}, x_{k-1}) + d(x_{k-1}, F(x_{k-2})) \\ &\leq \frac{1}{\lambda}d(x_k, F(x_{k-1})) + d(x_{k-1}, F(x_{k-2})) < \delta(\frac{1}{\lambda}+1). \end{aligned}$$

Hence,

$$x^{(1)} \in B(F(x_{k-2}), \delta(\frac{1}{\lambda}+1)) \subset B(F(x_{k-2}), \delta\frac{\lambda}{\lambda-1}) \subset F(B(x_{k-2}, \epsilon)),$$

and, as before, we can choose $x^{(2)} \in B(x_{k-2}, \epsilon)$ such that $F(x^{(2)}) = x^{(1)}$.

In this way at the k-th step we find $x = x^{(k)} \in X$ such that

$$d(F^i(x), x_i) = d(x^{(k-i)}, x_i) < \epsilon \ \text{ for } i = 0, \ldots k.$$

Since k is arbitrary, we can immediatly conclude the proof by contradiction. Q.E.D.

Theorem 1.2.10 [Sa] *If $F \in \mathcal{E}(X)$ then:*

(a) the set of fixed points of F on X,

$$\mathrm{Fix}_X(F) \overset{\mathrm{def}}{=} \{x \in X \ : \ F(x) = x\},$$

is non-empty and finite;

(b) the set of periodic points of F on X,

$$\mathrm{Per}_X(F) \overset{\mathrm{def}}{=} \bigcup_{k \geq 1} \mathrm{Fix}_X(F^k)$$

is countable and dense in X.

Proof. (a) Assume that F has infinite fixed points. Since X is compact, there exists a sequence of distinct fixed points $\{x_k\}_{k\geq 0}$ converging to a fixed point y. By (3), when $d(x_k, y) \leq \epsilon_0$ then

$$d(x_k, y) = d(F(x_k), F(y)) \geq \lambda d(x_k, y).$$

Hence $x_k = y$ for infinite k, that is a contradiction.

Now we prove that there exists at least one fixed point.

Let $\mathcal{V}$ be a cover of X consisting of a number $m \geq 1$ of open sets whose diameter is less than ϵ_1. Take $x_0 \in X$: if $F(x_0) = x_0$, then we have a fixed point. Otherwise, since X is connected, there is a *chain* of open sets $\{V_i \ : \ i = 1, \ldots, m'\} \subset \mathcal{V}$ with $1 \leq m' \leq m$ from $F(x_0)$ to x_0, that is, $F(x_0) \in V_0$, $x_0 \in V_{m'}$ and $V_i \cap V_{i+1} \neq \emptyset$ for $i = 0, \ldots, m' - 1$.

Now, by Theorem 1.2.7, if V is an open set such that $\mathrm{diam}(V) \leq \epsilon_1$ and $y \in F^{-1}(V)$ then there is a unique open set W such that $y \in W$, $\mathrm{diam}(W) \leq \frac{\epsilon_1}{\lambda}$ and W is mapped homeomorphically onto V. Applying this result, we can construct a new chain $\{V_i^1 \ : \ i = 1, \ldots, m'\}$ from x_0 to $x_1 \in F^{-1}(x_0)$ choosing each V_i^1 in $F^{-1}(V_i)$. Next, we lift this chain to a chain from x_1 to $x_2 \in F^{-1}(x_1)$.

Continuing this construction inductively we obtain a sequence $\{x_k\}_{k\geq 0}$ such that if $k > k'$ then

$$d(x_k, x_{k'}) \leq (\frac{1}{\lambda^k} + \ldots + \frac{1}{\lambda^{k'-1}}) m' \epsilon_1 < \frac{1}{\lambda^{k'}} \frac{m\epsilon_1}{\lambda - 1}.$$

Thus $\{x_k\}_{k\geq 0}$ is a Cauchy sequence which converges to some $y \in X$, because X is compact. Since $F(x_k) = x_{k-1}$ for $k \geq 1$, then

$$F(y) = F(\lim_{k\to\infty} x_k) = \lim_{k\to\infty} F(x_k) = \lim_{k\to\infty} x_{k-1} = y.$$

(b) Since also $F^k \in \mathcal{E}(X)$, then, by (a), $\mathrm{Fix}_X(F^k)$ is finite for all $k \geq 0$. Therefore the set of periodic points is countable.

By Theorem 1.2.5, F is positively expansive with some constant $c > 0$. Now, we show that whenever $0 < r < \frac{1}{2}c$ and for each $x_0 \in X$, there is $k \geq 1$ such that F^k has a fixed point in $B(x_0, r)$.

Choose $\delta > 0$ corresponding to $\epsilon = r$ in the definition of P.O.T.P., and take $k \geq 1$ such that $\delta\lambda^k > m\epsilon_1$, where m is the cardinality of the finite open cover $\mathcal{V}$ of (a).

If $F^k(x_0) \neq x_0$, following the procedure described in (a), we can costruct a set of points $\{x_0, x_1, \ldots, x_{k-1}\}$ such that $x_i \in F^{-1}(x_{i-1})$ for $i = 1, \ldots, k-1$ and the k-th chain goes from $F(x_0)$ to x_{k-1}. Therefore

$$d(x_{k-1}, F(x_0)) \leq m\frac{\epsilon_1}{\lambda^k} < \delta.$$

Hence, $\{x_0, x_{k-1}, x_{k-2}, \ldots, x_1, x_0, x_{k-1} \ldots\}$ is a δ-pseudo orbit. Since, by Theorem 1.2.9, F has P.O.P.T., there exists a r-tracing point $y \in B(x_0, r)$. Clearly also the point $F^k(y)$ is r-tracing.

Therefore $d(F^m(y), F^m(F^k(y)) \leq 2r < c$ for all $m \geq 0$ and, since F is positively expansive with constant c, this implies that $F^k(y) = y$. Q.E.D.

§1.3 Expanding maps on closed manifolds.

Let $\mathcal{M}$ be an n-dimensional connected compact topological manifold with boundary ∂M.

Theorem 1.3.1 [Hira2] *If $\mathcal{E}(\mathcal{M}) \neq \emptyset$ then $\partial M = \emptyset$.*

Proof. Let $F \in \mathcal{E}(\mathcal{M})$ then $F(\mathcal{M} \setminus \partial\mathcal{M}) \subset \mathcal{M} \setminus \partial M$ because F is a local homeomorphism. Let $x \in \mathcal{M}$ and let $0 < r < \frac{\epsilon}{2}$. We denote by $B_c(x,r)$ the connected component of x in $B(x,r)$ then

$$B_c(x,r) \subset \mathcal{M} \setminus \partial M \quad \text{implies} \quad F(B_c(x,r)) \supset B_c(F(x), \lambda r). \tag{6}$$

In fact, assume $y \in B_c(F(x), \lambda r) \setminus F(B_c(x,r))$. Let $\sigma : [0,1] \to B_c(F(x), \lambda r)$ be a path joining $\sigma(0) = F(x)$ and $\sigma(1) = y$. Since $\sigma(0) \in F(B_c(x,r))$ and $F(B_c(x,r))$ is open, there is $0 < t_0 < 1$ such that $\sigma([0,t_0[) \subset F(B_c(x,r))$ and $\sigma(t_0) \notin F(B_c(x,r))$. By the choice of r, the map $F_{|\overline{B_c(x,r)}}$ is a homeomorphism and, since $\sigma([0,t_0]) \subset \overline{F(B_c(x,r))} = F(\overline{B_c(x,r)})$, there is a path $\tilde{\sigma} : [0,t_0] \to \overline{B_c(x,r)}$ such that $F \circ \tilde{\sigma} = \sigma$. Hence, $\tilde{\sigma}(t_0) \notin B_c(x,r)$. On the other hand, since $\sigma(t_0) \in B_c(F(x), \lambda r)$, we have $\lambda d(\tilde{\sigma}(t_0), x) \leq d(\sigma(t_0), F(x)) < \lambda\epsilon$, that is $\tilde{\sigma}(t_0) \in B_c(x,r)$, thus a contradiction.

Now, we define for $r > 0$ the following set

$$\mathcal{M}(r) = \{x \in \mathcal{M} \setminus \partial M \ : \ B_c(x,r) \subset \mathcal{M} \setminus \partial M\}.$$

Since ∂M is a closed proper subset of $\mathcal{M}$, there is $0 < r_0 < \frac{\epsilon}{2}$ such that $\mathcal{M}(r_0) \neq \emptyset$. By (6), $F(\mathcal{M}(r_0)) \subset \mathcal{M}(\lambda r_0)$ and it is easy to check that for $1 < \alpha < \lambda$

$$F(\overline{\mathcal{M}(r_0)}) \subset \overline{F(\mathcal{M}(r_0))} \subset \overline{\mathcal{M}(\lambda r_0)} \subset \mathcal{M}(\alpha r_0) \subset \mathcal{M}(r_0). \tag{7}$$

The set $Y \overset{\text{def}}{=} \bigcap_{k \geq 0} F^k(\overline{\mathcal{M}(r_0)})$ is the intersection of a nested sequence of compact sets, therefore $Y \subset \mathcal{M}(r_0)$ is a non-empty closed set such that $f(Y) = Y$.

For $r > 0$, define the set $Y(r) = \bigcup_{y \in Y} B_c(y,r)$. Let $y \in Y$ and let $x \in B_c(y, (\alpha-1)r_0)$ for $1 < \alpha < \lambda$. Then $B_c(x, r_0) \subset B_c(y, \alpha r_0)$ and, since by (7) $Y \subset \mathcal{M}(\alpha r_0)$, we have that $B_c(y, \alpha r_0) \subset \mathcal{M} \setminus \partial\mathcal{M}$ and therefore $x \in \mathcal{M}(r_0)$. Hence $Y((\alpha-1)r_0) \subset \mathcal{M}(r_0)$ and, since $F(Y) = Y$,

$$Y \subset \bigcap_{k \geq 0} F^k(Y((\alpha-1)r_0)) \subset \bigcap_{k \geq 0} F^k(\overline{\mathcal{M}(r_0)}) = Y. \tag{8}$$

By (6), if $y \in Y$ then $F(B_c(y,(\alpha-1)r_0)) \supset B_c(F(y),\lambda(\alpha-1)r_0)$ and therefore

$$F(Y((\alpha-1)r_0)) \supset Y(\lambda(\alpha-1)r_0) \supset Y((\alpha-1)r_0).$$

Then, by (8), $Y = Y((\alpha-1)r_0))$, that is Y is open. Therefore $Y = \mathcal{M}$ because $\mathcal{M}$ is connected, and, since $Y \subset \mathcal{M}(r_0) \subset \mathcal{M} \setminus \partial\mathcal{M}$, the boundary $\partial\mathcal{M}$ is empty. Q.E.D.

So, let $\mathcal{M}$ be a *closed manifold*, i. e. an n-dimensional connected compact topological manifold without boundary. We will denote by $\widetilde{\mathcal{M}}$ the universal covering space of $\mathcal{M}$ and let $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ be the covering map. We denote by $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ the *deck transformation group* i. e. the set of all homeomorphisms γ of $\widetilde{\mathcal{M}}$ such that $\pi \circ \gamma = \pi$.

Given a continuous map $F : \mathcal{M} \to \mathcal{M}$, a lifting of F is a continuous map $\widetilde{F} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{F}} & \widetilde{\mathcal{M}} \\ \pi\downarrow & & \downarrow\pi \\ \mathcal{M} & \xrightarrow{F} & \mathcal{M} \end{array}.$$

There is always a lifting $\widetilde{F}$ of F, and all other liftings have the form $\gamma \circ \widetilde{F}$ with $\gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$. Moreover F induces a homomorphism $\widetilde{F}^\sharp$ of the group $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ such that

$$\widetilde{F} \circ \gamma = \widetilde{F}^\sharp(\gamma) \circ \widetilde{F} \qquad \forall \gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M}).$$

An open connected $C \subset \widetilde{\mathcal{M}}$ will be called a *covering domain* if $\overline{C}$ is compact and $\pi(C) = \mathcal{M}$.

Remark 1.3.2 Since $\widetilde{\mathcal{M}}$ is simply connected, $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ is isomorphic to the fundamental group $\pi_1(\mathcal{M}, x_0)$ for all $x_0 \in \mathcal{M}$. More precisely, for $[s] \in \pi_1(\mathcal{M}, x_0)$ then there is a unique $\gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ such that, given a representative $s : [0,1] \to \mathcal{M}$ of $[s]$, $\tilde{s}(1) = \gamma(\tilde{s}(0))$ for any lifting $\tilde{s}$ of s. Therefore, if $\gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ and $\gamma(y) = y$ for some $y \in \widetilde{\mathcal{M}}$ then $\gamma = e$ where

e is the identity element of $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$. Moreover, $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ is finitely generated because $\mathcal{M}$ is compact.

Theorem 1.3.3 [CoRe] *Let $F \in \mathcal{E}(\mathcal{M})$. Then the following statements hold:*

(a) there exists a complete compatible metric $\tilde{d}$ on the covering space $\widetilde{\mathcal{M}}$ such that

$$\tilde{d}(y, y') = d(\pi(y), \pi(y')) \text{ when } d(\pi(y), \pi(y')) \leq \epsilon_1 \text{ for } y, y' \in \widetilde{\mathcal{M}},$$

and each $\gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ is a $\tilde{d}$-isometry;

(b) each lifting $\widetilde{F}$ is an expanding homeomorphism of $\widetilde{\mathcal{M}}$ and

$$\tilde{d}(\widetilde{F}(y), \widetilde{F}(y')) \geq \lambda \tilde{d}(y, y') \qquad \forall y, y' \in \widetilde{\mathcal{M}} \tag{9}$$

hence $\widetilde{F}^{-1}$ is a contraction map and has a unique fixed point.

Proof. (a) The metric $\tilde{d}$ is defined as follows: for all $y, y' \in \widetilde{\mathcal{M}}$ let

$$\tilde{d}(y, y') \stackrel{\text{def}}{=} \inf\{\sum_{i=0}^{m} d(\pi(y_i), \pi(y_{i+1}))\}$$

where the infimum is taken over all the ϵ_1-*chains* from y to y', i. e. the finite sets $\{y_0, y_1, \ldots, y_m\}$ such that $y_0 = y$, $y_m = y'$ and $d(\pi(y_i), \pi(y_{i+1})) \leq \epsilon_1$ for $i = 0, \ldots, m-1$.

The set of (y, y') for which there exists at least a chain is non-empty, open and closed in the connected space $\widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}}$. Therefore $\tilde{d}$ is finite. By standard arguments, we can show that $\tilde{d}$ is a metric. Note that

$$d(\pi(y), \pi(y')) \leq \tilde{d}(y, y') \qquad \forall y, y' \in \widetilde{\mathcal{M}}$$

and the equality holds if $d(\pi(y), \pi(y')) \leq \epsilon_1$. Therefore, since d is complete and compatible with respect to the topology of $\mathcal{M}$, also the metric $\tilde{d}$ is complete and it is compatible with respect to the topology of $\widetilde{\mathcal{M}}$.

Each $\gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ is a $\tilde{d}$-isometry because $\pi = \pi \circ \gamma$.

(b) By Theorem 1.2.7, F is a self-covering map of $\mathcal{M}$. Thus $F \circ \pi$ is a covering map from $\widetilde{\mathcal{M}}$ to $\mathcal{M}$. Since $\widetilde{\mathcal{M}}$ is the universal covering space, the covering maps π and $F \circ \pi$ are equivalent, i. e. there is a homeomorphism h of $\widetilde{\mathcal{M}}$ such that $F \circ \pi = \pi \circ h$. Hence h is a lifting of F and this means that all the liftings of F are homeomorphisms $\widetilde{\mathcal{M}}$.

Let $y, y' \in \widetilde{\mathcal{M}}$ and choose a $\frac{\epsilon_1}{\lambda}$-chain $\{y_0, y_1, \ldots, y_m\}$ from $\widetilde{F}(y)$ to $\widetilde{F}(y')$ then $\{\widetilde{F}^{-1}(y_0), \widetilde{F}^{-1}(y_1), \ldots, \widetilde{F}^{-1}(y_m)\}$ is a chain from y to y'. In fact, for $i = 0, \ldots, m-1$ by (c) of Theorem 1.2.7,

$$d(\pi(\widetilde{F}^{-1}(y_i)), \pi(\widetilde{F}^{-1}(y_{i+1})) \leq \frac{\epsilon_1}{\lambda}.$$

Therefore, by (3) ,

$$d(F(\pi(\widetilde{F}^{-1}(y_i))), F(\pi(\widetilde{F}^{-1}(y_{i+1}))) \geq \lambda d(\pi(\widetilde{F}^{-1}(y_i)), \pi(\widetilde{F}^{-1}(y_{i+1})).$$

Since $F(\pi(\widetilde{F}^{-1}(y_i))) = \pi(y_i)$, summing over i we obtain

$$\sum_{i=0}^{m-1} d(\pi(y_i), \pi(y_{i+1})) \geq \lambda \tilde{d}(y, y').$$

Then taking the infimum over the chains from $\widetilde{F}(y)$ to $\widetilde{F}(y')$ we have (9). Q.E.D.

The previous theorem yields the following properties:

Proposition 1.3.4 [Sh1] *If $F \in \mathcal{E}(\mathcal{M})$ then*

(a) if $r > 0$, there exists an integer $k_r \geq 0$ such that $F^{k_r}(B(x, r)) = \mathcal{M}$ for all $x \in \mathcal{M}$;

(b) the set $\bigcup_{k \geq 0} F^{-k}(x)$ is a dense in $\mathcal{M}$ for all $x \in \mathcal{M}$;

(c) the set

$$\mathrm{Do}_{\mathcal{M}}(F) \stackrel{\text{def}}{=} \{x \in \mathcal{M} \ : \ \{F^k(x)\}_{k \geq 0} \textit{ is dense in } \mathcal{M}\}$$

is dense in $\mathcal{M}$;

(d) the Euler characteristic $\chi(\mathcal{M}) = 0$.

Proof. (a) We can assume that $r < \epsilon_1$. Let C be a covering domain and take $k_r \geq 0$ such that $\lambda^{k_r} r \geq \mathrm{diam}(C)$. Let $y \in C \cap \pi^{-1}(x)$ then, by (a) of Theorem 1.3.3, $\widetilde{B}(y,r) \subset \pi^{-1}(B(x,r))$. Take $\gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ such that $\widetilde{F}^{k_r}(y) \in \gamma(C)$. Thus, by (b) of Theorem 1.3.3,

$$\widetilde{F}^{k_r}(\widetilde{B}(y,r)) \supset \widetilde{B}(\widetilde{F}^{k_r}(y), \lambda^{k_r} r) \supset \gamma(C). \tag{10}$$

Therefore the proof is complete because by (10)

$$\begin{aligned} F^{k_r}(B(x,r)) &\supset F^{k_r}(\pi(\pi^{-1}(B(x,r)))) = \pi(\widetilde{F}^{k_r}(\widetilde{B}(y,r)) \\ &\supset \pi(\gamma(C)) = \pi(C) = \mathcal{M}. \end{aligned}$$

(b) If V is an open subset of $\mathcal{M}$ then, by (a), $\bigcup_{k\geq 0} F^k(V) = \mathcal{M}$.

If $x \in \mathcal{M}$ then there exists $k_0 \geq 0$ such that $x \in F^{k_0}(V)$, that is, $F^{-k_0}(x) \cap V \neq \emptyset$. Thus $\bigcup_{k\geq 0} F^{-k}(x)$ is dense in $\mathcal{M}$.

(c) If $\{V_i\}_{i\geq 1}$ is a countable base of open sets for the topology on $\mathcal{M}$ then

$$\mathrm{Do}_{\mathcal{M}}(F) = \bigcap_{i=1}^{\infty} (\bigcup_{k=0}^{\infty} F^{-k}(V_n)).$$

Let $U \subset \mathcal{M}$ be a non-empty open set. Then

$$\mathrm{Do}_{\mathcal{M}}(F) \cap U = \bigcap_{i=1}^{\infty} U_i \ \text{ with } \ U_i = \bigcup_{k=0}^{\infty} F^{-k}(V_i) \cap U.$$

If U' is an open subset of U then, by (a), $F^k(U') \cap V_n \neq \emptyset$ for some $k \geq 0$; therefore

$$U_i \cap U' = \bigcup_{k=0}^{\infty} F^{-k}(V_i) \cap U' \neq \emptyset.$$

Hence, $\mathrm{Do}_{\mathcal{M}}(F) \cap U$ is the intersection of a countable collection of non-empty open sets U_i which are dense in $\overline{U}$.

Since $\overline{U}$ is compact, it is a complete metric space and by Baire theorem, $\mathrm{Do}_{\mathcal{M}}(F) \cap \mathcal{M}$ is non-empty.

(d) Since F is a self-covering of $\mathcal{M}$ with degree $N \geq 2$, then $\chi(\mathcal{M}) = N\chi(\mathcal{M})$ and therefore $\chi(\mathcal{M}) = 0$. Q.E.D.

Remark 1.3.5 The previous proposition and Theorem 1.2.10 imply that the expanding maps are *chaotic* according to the definition of Devaney (see [BBCDS]). In fact, $F \in \mathcal{E}(\mathcal{M})$ implies that the continuous map F has the following two properties:

(a) the map F is *one-sided topologically transitive*, i. e. there exists a point $x \in \mathcal{M}$ whose forward orbit $\{F^k(x)\}_{k\geq 0}$ is dense in $\mathcal{M}$;

(b) the set $\mathrm{Per}_{\mathcal{M}}(F)$ is dense in $\mathcal{M}$.

Note that, if X is any compact, connected, metrizable space, then there exist expanding maps which are not chaotic.

In fact let $T_1 \stackrel{\text{def}}{=} \{z \in \mathbb{C} \ : \ |z-1| = 1\}$, $T_{-1} \stackrel{\text{def}}{=} \{z \in \mathbb{C} \ : \ |z+1| = 1\}$ and $X \stackrel{\text{def}}{=} T_1 \cup T_{-1}$ and define the map

$$F(z) \stackrel{\text{def}}{=} \begin{cases} -(z-1)^2+1 & \text{if } z \in T_1 \\ (z+1)^2-1 & \text{if } z \in T_{-1} \end{cases} .$$

Therefore, $F \in \mathcal{E}(X)$ because $F_{T_i} \in \mathcal{E}(T_i)$ for $i = 1, -1$ and $F(0) = 0$. Since $F(T_i) = T_i$ for $i = 1, -1$, there is no $x \in X$ whose forward orbit is dense in X.

Let $\mathcal{M}$, $\mathcal{M}'$ be two closed manifolds. If $\psi : \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M}) \to \mathcal{D}(\widetilde{\mathcal{M}}'/\mathcal{M}')$ is a homomorphism, we denote by $\mathcal{S}_\psi$ the set of all continuous maps $\tilde{\alpha} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$ which are liftings of continuous maps $\alpha : \mathcal{M} \to \mathcal{M}'$ such that $\tilde{\alpha}^\sharp = \psi$.

Theorem 1.3.6 [Sh1] *Let $F \in \mathcal{E}(\mathcal{M})$, $\Phi \in \mathcal{E}(\mathcal{M}')$ and $\psi : \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M}) \to \mathcal{D}(\widetilde{\mathcal{M}}'/\mathcal{M}')$ be a isomorphism. If $\psi \circ \widetilde{F}^\sharp = \widetilde{\Phi}^\sharp \circ \psi$, then there exists a unique $\tilde{\alpha}_0 \in \mathcal{S}_\psi$ such that*

$$\tilde{\alpha}_0 \circ \widetilde{F} = \widetilde{\Phi} \circ \tilde{\alpha}_0.$$

Moreover, $\alpha_0 : \mathcal{M} \to \mathcal{M}'$ is a homeomorphism and F and Φ are topologically conjugate: $\alpha_0 \circ F = \Phi \circ \alpha_0$.

Proof. Assume that Φ is expanding with respect to the metric d' on $\mathcal{M}'$. Then, by Theorem 1.3.3, there is a metric $\tilde{d}'$ on $\widetilde{\mathcal{M}}'$ for which the elements of $\mathcal{D}(\widetilde{\mathcal{M}}'/\mathcal{M}')$ are isometries and $\tilde{\Phi}^{-1}$ is a $\frac{1}{\lambda_\Phi}$-contraction map, where $\lambda_\Phi > 1$ is the expanding constant of Φ. Now, define the map $D : \mathcal{S}_\psi \times \mathcal{S}_\psi \to \mathbb{R}^+$:

$$D(\tilde{\alpha}, \tilde{\beta}) \stackrel{\text{def}}{=} \sup_{y \in \widetilde{\mathcal{M}}} \{\tilde{d}'(\tilde{\alpha}(y), \tilde{\beta}(y))\} \qquad \forall \tilde{\alpha}, \tilde{\beta} \in \mathcal{S}_\psi.$$

To prove that the map D is a distance, it suffices to show that D is finite on $\mathcal{S}_\psi$. Let C be a covering domain of $\mathcal{M}$ and take $\tilde{\alpha}, \tilde{\beta} \in \mathcal{S}_\psi$. If $y \in \widetilde{\mathcal{M}}$, then there exists $\gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ such that $\gamma(y) \in C$. Thus, since $\psi(\gamma)$ is a $\tilde{d}'$-isometry,

$$\tilde{d}'(\tilde{\alpha}(y), \tilde{\beta}(y)) = \tilde{d}'(\psi(\gamma)(\tilde{\alpha}(y)), \psi(\gamma)(\tilde{\beta}(y))) = \tilde{d}'(\tilde{\alpha}(\gamma(y)), \tilde{\beta}(\gamma(y))),$$

and $D(\tilde{\alpha}, \tilde{\beta}) = \sup_{y \in \overline{C}} \{\tilde{d}'(\tilde{\alpha}(y), \tilde{\beta}(y))\} < +\infty$ because $\overline{C}$ is compact.

The distance D is complete: if $\{\tilde{\alpha}_i\}$ is a Cauchy sequence in $\mathcal{S}_\psi$ then $\tilde{\alpha}_i$ converges to a continuous map $\tilde{\alpha}$. Moreover, if $\gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ then, since $\tilde{\alpha}_i \circ \gamma = \psi(\gamma) \circ \tilde{\alpha}_i$ for all $i \geq 1$, also $\tilde{\alpha} \circ \gamma = \psi(\gamma) \circ \tilde{\alpha}$, i. e. $\alpha \in \mathcal{S}_\psi$.

For $\tilde{\alpha} \in \mathcal{S}_\psi$ define $\mathcal{F}(\tilde{\alpha}) = \tilde{\Phi}^{-1} \circ \tilde{\alpha} \circ \widetilde{F}$. Now we prove that $\mathcal{F}$ is a contraction map in $\mathcal{S}_\psi$ with respect to the distance D. Thus, since D is complete, $\mathcal{F}$ has a unique fixed point $\tilde{\alpha}_0 \in \mathcal{S}_\psi$ such that

$$\tilde{\alpha}_0 \circ \widetilde{F} = \tilde{\Phi} \circ \tilde{\alpha}_0.$$

First we prove that $\mathcal{F}(\mathcal{S}_\psi) \subset \mathcal{S}_\psi$. In fact, let $\gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$. Then

$$\begin{aligned}
\mathcal{F}(\tilde{\alpha}) \circ \gamma &= \tilde{\Phi}^{-1} \circ \tilde{\alpha} \circ \widetilde{F} \circ \gamma = \tilde{\Phi}^{-1} \circ \tilde{\alpha} \circ \widetilde{F}^\sharp(\gamma) \circ \widetilde{F} \\
&= \tilde{\Phi}^{-1} \circ \psi(\widetilde{F}^\sharp(\gamma)) \circ \tilde{\alpha} \circ \widetilde{F} \\
&= (\tilde{\Phi}^{-1})^\sharp(\psi(\widetilde{F}^\sharp(\gamma))) \circ \tilde{\Phi}^{-1} \circ \tilde{\alpha} \circ \widetilde{F} \\
&= \psi(\gamma) \circ \mathcal{F}(\tilde{\alpha})
\end{aligned}$$

because, by hypothesis, $\psi \circ \widetilde{F}^\sharp = \tilde{\Phi}^\sharp \circ \psi$.

The map $\mathcal{F}$ is a contraction:

$$D(\mathcal{F}(\tilde{\alpha}), \mathcal{F}(\tilde{\beta})) = \sup_{y \in \widetilde{\mathcal{M}}} \{\tilde{d}'(\tilde{\Phi}^{-1} \circ \tilde{\alpha} \circ \widetilde{F}(y), \tilde{\Phi}^{-1} \circ \tilde{\beta} \circ \widetilde{F}(y))\}$$

$$\begin{aligned} &\leq \frac{1}{\lambda_\Phi} \sup_{y\in\widetilde{\mathcal{M}}}\{\tilde{d}'(\tilde{\alpha}\circ\widetilde{F}(y),\tilde{\beta}\circ\widetilde{F}(y))\} \\ &\leq \frac{1}{\lambda_\Phi} \sup_{y\in\widetilde{\mathcal{M}}}\{\tilde{d}'(\tilde{\alpha}(y)),\tilde{\beta}(y))\} \leq \frac{1}{\lambda_\Phi} D(\tilde{\alpha},\tilde{\beta}). \end{aligned}$$

Now, changing the role of F and Φ, we can find in the same way a map $\tilde{\alpha}_1 \in \mathcal{S}_{\psi^{-1}}$ such that:

$$\tilde{\alpha}_1 \circ \widetilde{\Phi} = \widetilde{F} \circ \tilde{\alpha}_1.$$

Hence

$$\begin{cases} (\tilde{\alpha}_0 \circ \tilde{\alpha}_1) \circ \widetilde{\Phi} = \tilde{\alpha}_0 \circ \widetilde{F} \circ \tilde{\alpha}_1 = \widetilde{\Phi} \circ (\tilde{\alpha}_0 \circ \tilde{\alpha}_1) \\ (\tilde{\alpha}_1 \circ \tilde{\alpha}_0) \circ \widetilde{F} = \tilde{\alpha}_1 \circ \widetilde{\Phi} \circ \tilde{\alpha}_0 = \widetilde{F} \circ (\tilde{\alpha}_1 \circ \tilde{\alpha}_0) \end{cases}.$$

Now, $\tilde{\alpha}_0 \circ \tilde{\alpha}_1 \in \mathcal{S}_{\mathrm{Id}'}$ and $\tilde{\alpha}_1 \circ \tilde{\alpha}_0 \in \mathcal{S}_{\mathrm{Id}}$ where Id and Id$'$ are the identity automorphisms respectively of $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ and $\mathcal{D}(\widetilde{\mathcal{M}}'/\mathcal{M}')$. By the first part of the proof, the equations

$$\tilde{\alpha} \circ \widetilde{\Phi} = \widetilde{\Phi} \circ \tilde{\alpha} \ \text{ and } \ \tilde{\alpha} \circ \widetilde{F} = \widetilde{F} \circ \tilde{\alpha}$$

have a unique solution respectively in $\mathcal{S}_{\mathrm{Id}'}$ and $\mathcal{S}_{\mathrm{Id}}$. Thus $\tilde{\alpha}_0 \circ \tilde{\alpha}_1 = \mathrm{Id}'$, $\tilde{\alpha}_1 \circ \tilde{\alpha}_0 = \mathrm{Id}$ and $\tilde{\alpha}_0 : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$ is a homeomorphism. To verify that also α_0 is a homeomorphism it suffices to show that α_0 is injective. In fact, let $x_1, x_2 \in \mathcal{M}$ be such that $\alpha_0(x_1) = \alpha_0(x_2)$ and let $y_i \in \pi^{-1}(x_i)$ for $i = 1, 2$. Thus, there exists $\gamma' \in \mathcal{D}(\widetilde{\mathcal{M}}'/\mathcal{M}')$ such that $\gamma'(\tilde{\alpha}_0(y_1)) = \tilde{\alpha}_0(y_2)$. Since ψ is an isomorphism, if $\gamma = \psi^{-1}(\gamma')$, then

$$\tilde{\alpha}_0(\gamma(y_1)) = \psi(\gamma)(\tilde{\alpha}_0(y_1)) = \gamma'(\tilde{\alpha}_0(y_1)) = \tilde{\alpha}_0(y_2),$$

that is $x_1 = \pi(y_1) = \pi(\gamma(y_1)) = \pi(y_2) = x_2$. Moreover, the following equalities hold

$$\alpha_0 \circ F \circ \pi = \pi \circ \tilde{\alpha}_0 \circ \widetilde{F} = \pi \circ \widetilde{\Phi} \circ \tilde{\alpha}_0 = \Phi \circ \alpha_0 \circ \pi.$$

Q.E.D.

Remark 1.3.7 In the previous theorem, if $\Phi : \mathcal{M}' \to \mathcal{M}'$ is a continuous map, without assuming it to be expanding, then, there exists a continuous surjective map $\tilde{\alpha}_1 \in \mathcal{S}_{\psi^{-1}}$ such that: $\tilde{\alpha}_1 \circ \widetilde{\Phi} = \widetilde{F} \circ \tilde{\alpha}_1$.

Corollary 1.3.8 [Sh1] *Let $F, \Phi \in \mathcal{E}(\mathcal{M})$. If F and Φ are homotopic, then they are topologically conjugate.*

Proof. By the previous theorem it is enough to show that it is possible to choose two liftings $\widetilde{F}$ and $\widetilde{\Phi}$ such that $\widetilde{F}^\sharp = \widetilde{\Phi}^\sharp$. Let $x_0 \in \mathcal{M}$ and $y_0 \in \pi^{-1}(x_0)$. Since F and Φ are homotopic, there is a path $s : [0,1] \to \mathcal{M}$ from $F(x_0)$ to $\Phi(x_0)$ such that

$$F_\sharp([\sigma]) = s * \Phi_\sharp([\sigma]) * s^{-1} \qquad \forall [\sigma] \in \pi_1(\mathcal{M}, x_0) \tag{11}$$

where $F_\sharp$ and $\Phi_\sharp$ are the homomorphism induced respectively by F and Φ on $\pi_1(\mathcal{M}, x_0)$ (see for example [Sp] p. 52). Let $\widetilde{F}$ be a lifting of F and take the lifting $\tilde{s}_1$ such that $\tilde{s}_1(0) = \widetilde{F}(y_0)$, and the lifting $\widetilde{\Phi}$ such that $\widetilde{\Phi}(y_0) = \tilde{s}_1(1)$.

Now we show that this is a good choice of liftings for our purpose. If $\gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M}) \simeq \pi_1(\mathcal{M}, x_0)$, then there is a closed path σ at x_0 such that $\tilde{\sigma}(1) = \gamma(\tilde{\sigma}(0))$ for all liftings $\tilde{\sigma}$. Let $\tilde{\sigma}_0$ be the lifting starting from y_0 and define $\tilde{s}_2 = \widetilde{\Phi}^\sharp(\gamma) \circ \tilde{s}_1$, another lifting of s_2.

By (11), the paths $F_\sharp(\sigma)$ and $s * \Phi_\sharp(\sigma) * s^{-1}$ are homotopic and the paths $\widetilde{F} \circ \tilde{\sigma}_0$ and $\tilde{s}_1 * \widetilde{\Phi} \circ \tilde{\sigma}_0 * \tilde{s}_2^{-1}$ are their liftings. These liftings have the same starting point $\widetilde{F}(y_0)$ and, by the monodromy theorem, they must have also the same ending point. Hence

$$\begin{cases} (\widetilde{F} \circ \tilde{\sigma}_0)(1) = \widetilde{F}(\gamma(y_0)) = \widetilde{F}^\sharp(\gamma)(\widetilde{F}(y_0)) \\ (\tilde{s}_1 * \widetilde{\Phi} \circ \tilde{\sigma}_0 * \tilde{s}_2^{-1})(1) = \tilde{s}_2^{-1}(1) = \tilde{s}_2(0) = \widetilde{\Phi}^\sharp(\gamma)(\widetilde{F}(y_0)) \end{cases},$$

and thus, by Remark 1.3.2, $\widetilde{F}^\sharp(\gamma) = \widetilde{\Phi}^\sharp(\gamma)$. Q.E.D.

§1.4 Conjugation to infra-nil-manifold endomorphisms.

Theorem 1.4.1 [Sh1] *If $\mathcal{E}(\mathcal{M}) \neq \emptyset$ then:*

(a) $\mathcal{M}$ is aspherical, that is the homotopy groups $\pi_i(\mathcal{M}) = 0$ for $i \geq 2$;

(b) $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ is torsion free, i. e. $\gamma^k = e$, for some $\gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ and $k \geq 1$, implies $\gamma = e$.

Proof. (a) Since $\widetilde{\mathcal{M}}$ is the universal covering of $\mathcal{M}$ then $\pi_i(\mathcal{M}) \simeq \pi_i(\widetilde{\mathcal{M}})$ for $i \geq 2$. Therefore it suffices to prove that the homotopy groups $\pi_i(\widetilde{\mathcal{M}}) = 0$ for all $i \geq 1$. By (9), for all $k \geq 0$ and $r > 0$

$$\widetilde{F}^k(\widetilde{B}(y,r)) \supset \widetilde{B}(\widetilde{F}^k(y), \lambda^k r). \tag{12}$$

Let D^n be the closed unit disk of $\mathbb{R}^n$ and take an embedding $j : D^n \to \widetilde{\mathcal{M}}$ such that $j(0) = y_0$, the unique fixed point of $\widetilde{F}$. Let $W_k = \widetilde{F}^k(j(D^n))$ for $k \geq 1$ then, by (12), it follows immediatly that

$$W_k \subset W_{k+1} \quad \forall k \geq 1 \text{ and } \bigcup_{k \geq 1} W_k = \widetilde{\mathcal{M}}.$$

Let $\xi : S^i \to \mathcal{M}$ be a continuous map, where S^i is the unit sphere of $\mathbb{R}^{i+1}$ for $i \geq 1$. Then, there is some $k \geq 1$ such that $\xi(S^i) \subset W_k$. Moreover, the homotopy group $\pi_i(W_k) = 0$ because W_k is homeomorphic to D^n. Therefore ξ is homotopically equivalent to 0 in $\pi_i(W_k)$ and, since $W_k \subset \widetilde{\mathcal{M}}$, ξ is homotopically equivalent to 0 also in $\pi_i(\widetilde{\mathcal{M}})$.

(b) Assume by contradiction that $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ contains a non-trivial finite cyclic subgroup of order $k \geq 2$. Since $\mathcal{M}$ has universal covering $\widetilde{\mathcal{M}}$ and $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M}) \simeq \pi_1(\mathcal{M})$, there exists a covering space $\widehat{\mathcal{M}}$ of $\mathcal{M}$ such that $\pi_1(\widehat{\mathcal{M}}) \simeq \mathbb{Z}_k$.

By (a), also $\widehat{\mathcal{M}}$ is aspherical and therefore the integral homology groups

$$H_i(\widehat{\mathcal{M}}, \mathbb{Z}) \simeq \begin{cases} 0 & i \text{ even, } i > 0 \\ \mathbb{Z} & i = 0 \\ \mathbb{Z}_k & i \text{ odd, } i > 0 \end{cases}$$

(see for example [Sp] p. 503). But $\widehat{\mathcal{M}}$ is finite-dimensional, thus $H_i(\widehat{\mathcal{M}}, \mathbb{Z}) = 0$ for $i > n = \dim(\mathcal{M})$, hence a contradiction. Q.E.D.

Consider a group Γ generated by $\gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma$. Each element $\gamma \in \Gamma$ can be represented by a word $\gamma_{i_1}^{p_1}\gamma_{i_2}^{p_2}\cdots\gamma_{i_l}^{p_l}$ with $p_1, p_2, \ldots, p_l \in \mathbb{Z}$, and the integer $|p_1| + |p_2| + \cdots + |p_l|$ is called the *length* of the word.

The *norm* $\|\gamma\|$ relative to the set of generators $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ is defined as the minimal length of the words representing γ. This norm has the following elementary properties

$$\|\gamma\| = \|\gamma^{-1}\| \text{ and } \|\gamma\gamma'\| \leq \|\gamma\| + \|\gamma'\| \qquad \forall \gamma, \gamma' \in \Gamma.$$

Moreover, if $\|\cdot\|'$ is another norm corresponding to a different set of generators of Γ then there exists a positive costant c such that

$$\frac{1}{c}\|\gamma\| \leq \|\gamma\|' \leq c\|\gamma\| \qquad \forall \gamma \in \Gamma. \tag{13}$$

The norm $\|\cdot\|$ provides Γ with a left invariant metric $d_\Gamma(\gamma_1, \gamma_2) \stackrel{\text{def}}{=} \|\gamma_1^{-1}\gamma_2\|$, and we denote by $B_\Gamma(\gamma_0, r) \stackrel{\text{def}}{=} \{\gamma \in \Gamma \,:\, d_\Gamma(\gamma_0, \gamma) < r\}$ the open ball of radius $r > 0$ centered at $\gamma_0 \in \Gamma$.

Definition 1.4.2 A group Γ with generators $\gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma$ has *polynomial growth* if there is a real polynomial p such that

$$\operatorname{card}(B_\Gamma(e, r)) \leq p(r) \qquad \forall r \geq 1$$

where e is the identity element of Γ. By (13), this property does not depend on the particular choice of the set of generators.

The following theorem is the first step towards the Conjugation Theorem.

Theorem 1.4.3 [Fr] *If $\mathcal{E}(\mathcal{M}) \neq \emptyset$ then $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ has polynomial growth.*

Proof. Let $F \in \mathcal{E}(\mathcal{M})$ with expanding constant $\lambda > 1$. Replacing F by F^{k_0} for some $k_0 \geq 1$, by Proposition 1.2.3, we can assume that $\lambda \geq 2$. Choose a set of generators $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ of $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ and let $y_0 \in \widetilde{\mathcal{M}}$ be the unique fixed point of the lifting $\widetilde{F}$. Define $R \stackrel{\text{def}}{=} \max_{i=1,\ldots,m}\{\tilde{d}(y_0, \gamma_i(y_0))\}$ and $\mathcal{A} \stackrel{\text{def}}{=} \{\gamma(y_0) \,:\, \gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})\} \subset \widetilde{\mathcal{M}}$.

Now, if $\gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ with $\|\gamma\| < r$ then it is possible to write $\gamma = \gamma_{i_1}^{p_1}\gamma_{i_2}^{p_2}\cdots\gamma_{i_l}^{p_l}$ with $|p_1|+|p_2|+\cdots+|p_l| = \|\gamma\|$. Since the elements of $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ are $\tilde{d}$-isometries, then

$$\tilde{d}(y_0, \gamma(y_0)) \leq \sum_{k=1}^{l} |p_k|\tilde{d}(y_0, \gamma_{i_k}(y_0)) \leq \|\gamma\|R \leq rR.$$

Therefore

$$\operatorname{card}(B_\Gamma(e,r)) \leq \operatorname{card}(\mathcal{A} \cap \widetilde{B}(y_0, rR)) \qquad \forall r \geq 1. \tag{14}$$

Now, we are looking for an estimate on $\operatorname{card}(\mathcal{A} \cap \widetilde{B}(y_0, rR))$.

By (b) of Theorem 1.3.3,

$$\tilde{d}(\widetilde{F}^k(y), \widetilde{F}^k(y')) \geq \lambda^k \tilde{d}(y,y') \geq 2^k \tilde{d}(y,y') \quad \forall k \geq 1 \text{ and } \forall y, y' \in \widetilde{\mathcal{M}}.$$

Choose an integer $p \geq 2$ such that $\widetilde{B}(y_0, pR)$ is a covering domain and let

$$q \overset{\text{def}}{=} \max_{x \in \mathcal{M}} \{\operatorname{card}(\pi^{-1}(x) \cap \widetilde{B}(y_0, pR))\} < \infty.$$

Since $\widetilde{F}$ is a homeomorphism and the degree of F is $N \geq 2$, then, for $k \geq 1$,

$$\operatorname{card}(\mathcal{A} \cap \widetilde{F}^k(\widetilde{B}(y_0, pR))) = \operatorname{card}(\pi^{-1}(F^{-k}(\pi(y_0))) \cap \widetilde{B}(y_0, pR)) \leq qN^k,$$

because $\mathcal{A} = \pi^{-1}(\pi(y_0))$ and $\operatorname{card}(F^{-k}(\pi(y_0))\} = N^k$. Thus, by (14), for $k \geq 1$

$$\operatorname{card}(B_\Gamma(e, 2^k p)) \leq \operatorname{card}(\mathcal{A} \cap \widetilde{B}(y_0, 2^k pR)) \leq \operatorname{card}(\widetilde{F}^k(\mathcal{A} \cap \widetilde{B}(y_0, pR))) \leq qN^k.$$

Choose an integer d such that $2^d \geq N$ and $p^d \geq q$. Then, for $k \geq 1$,

$$\operatorname{card}(B_\Gamma(e, 2^k p)) \leq qN^k \leq (2^k p)^d.$$

If $r \geq 2^d p$ then $2^k p \leq r \leq 2^{k+1} p$ for some $k \geq d$ and

$$\operatorname{card}(B_\Gamma(e,r)) \leq \operatorname{card}(B_\Gamma(e, 2^{k+1}p)) \leq (2^{k+1}p)^d \leq 2^d r^d \leq r^{d+1}.$$

So, $\operatorname{card}(B_\Gamma(e,r)) \leq r^{d+1} + c_0$, where $c_0 \overset{\text{def}}{=} \max_{r<2^d}\{\operatorname{card}(B_\Gamma(e,r))\}$. Q.E.D.

Let $\mathcal{M}$ be a smooth closed manifold. Now, we define a special subclass of expanding maps.

Definition 1.4.4 A C^1-map $F : \mathcal{M} \to \mathcal{M}$ is called *differentiably expanding* if there are a compatible riemannian metric $||\cdot||$ on $T\mathcal{M}$ and a constant $\lambda > 1$ such that at any point $x \in \mathcal{M}$

$$||(D_x F)v|| \geq \lambda ||v|| \qquad \forall v \in T_x\mathcal{M}. \tag{15}$$

We will denote by $\mathcal{E}^1(\mathcal{M})$ the set of all differentiably expanding maps on $\mathcal{M}$.

Remark 1.4.5 The differentiably expanding maps are clearly expanding for the riemannian metric d. In fact, F is a local diffeomorphism and by Theorem 1.1.1, it is a self-covering. Moreover, there is $\epsilon_0 > 0$ such that $\epsilon_0 < \rho_0$ and

$$d(x, x') \leq \epsilon_0 \quad \text{implies} \quad d(F(x), F(x') \leq \eta_0,$$

where η_0 and ρ_0 are constants determined in Theorem 1.1.1. Let $d(x, x') \leq \epsilon_0$ and let $s : [0, 1] \to \mathcal{M}$ be a differentiable arc joining $F(x)$ and $F(x')$, which is minimal with respect to the metric d. If $D = s([0, 1])$ then $\mathrm{diam}(D) \leq \eta_0$ and Theorem 1.1.1, $(F_{|D})^{-1} \circ s$ is a differentiable arc joining x and x'. Therefore, by (15),

$$d(x, x') \leq \int_0^1 ||(D_{s(t)}F)^{-1} D_t s|| dt \leq \frac{1}{\lambda} \int_0^1 ||D_t s|| dt = \frac{1}{\lambda} d(F(x), F(x')).$$

On the other hand, there exist C^1-maps which are expanding but not differentiably expanding (see [CoRe]).

In fact let $\widetilde{F} : \mathbb{R} \to \mathbb{R}$ be a C^1-map with the following properties:

(a) $\widetilde{F}(0) = 0$ and $\widetilde{F}(1) = 2$;

(b) $D_0\widetilde{F} = D_1\widetilde{F} = 1$ and $D_y\widetilde{F} > 1$ whenever $y \in [0, 1]$;

(c) $\widetilde{F}(y + k) = \widetilde{F}(y) + 2k$ for all $y \in \mathbb{R}$ and $k \in \mathbb{Z}$.

For example, we can choose $\widetilde{F}(y) \stackrel{\text{def}}{=} y + y^2(y - 2)^2$ on $[0, 1]$ and extend it to all of $\mathbb{R}$ by (c).

Let F be the C^1-map induced by $\widetilde{F}$ on the circle $\mathbf{T}^1$, then since $D_x F = 1$ at the fixed point $x = \pi(0)$, $F \notin \mathcal{E}^1(\mathbf{T}^1)$ whereas it is open and locally expansive. Hence, by Theorem 1.2.5, F is an expandig map.

Proposition 1.4.6 *Let $F : \mathcal{M} \to \mathcal{M}$ be a C^1-map and $k \geq 1$. The map $F \in \mathcal{E}^1(\mathcal{M})$ iff $F^k \in \mathcal{E}^1(\mathcal{M})$.*

Proof. If $F \in \mathcal{E}^1(\mathcal{M})$ then obviously $F^k \in \mathcal{E}^1(\mathcal{M})$ with respect to the same metric and expanding constant $\lambda' \stackrel{\text{def}}{=} \lambda^k > 1$.

Assume now that $F^k \in \mathcal{E}^1(\mathcal{M})$ with respect to the metric $||\cdot||'$ and the costant $\lambda' > 1$. Then we may define the riemannian metric $||\cdot||$ on $T_x\mathcal{M}$ for all $x \in \mathcal{M}$ as follows:

$$||v||^2 \stackrel{\text{def}}{=} \sum_{i=1}^{k} ||(D_xF^{i-1})v||'^2 \qquad \forall v \in T_x\mathcal{M}.$$

Since $\mathcal{M}$ is compact and F is a C^1-map, there exists a costant $C > 0$ such that

$$||v||'^2 \leq ||v||^2 \leq C||v||'^2 \qquad \forall v \in T_x\mathcal{M},$$

i.e. the riemannian metrics $||\cdot||$ and $||\cdot||'$ are equivalent. Moreover,

$$\begin{aligned} ||(D_xF)v||^2 &= ||(D_xF)v||'^2 + \cdots + ||(D_{F(x)}F^{k-2})(D_xF)v||'^2 \\ &\quad + ||(D_{F(x)}F^{k-1})(D_xF)v||'^2 \\ &= ||(D_xF)v||'^2 + \cdots + ||(D_xF^{k-1})v||'^2 + ||(D_xF^k)v||'^2 \\ &\geq ||v||^2 - ||v||'^2 + \lambda'^2||v||'^2 \\ &\geq (1 + \frac{\lambda'^2 - 1}{C})||v||^2 \geq \lambda||v||^2 \end{aligned}$$

and $F \in \mathcal{E}^1(\mathcal{M})$ with expanding constant $\lambda \stackrel{\text{def}}{=} 1 + \frac{\lambda'^2 - 1}{C}$. Q.E.D.

Let G be a group. A finite *normal series*

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_l = \{e\}$$

where each G_i is a subgroup of G_{i-1} and it is normal in G (e is the identity element of G), is said to be *central* if G_i is a subgroup of the commutator group $[G_{i-1}, G]$ for $i = 1, \ldots, l$. A group admitting a central series is called *nilpotent*, and the length of the group's shortest central series is termed its *nilpotency class*. Note that the non-trivial abelian groups are just the nilpotent groups of class one.

Let N be a connected, simply connected, nilpotent Lie group with a left invariant riemannian metric and let $\mathrm{Aut}(\mathsf{N})$ be the group of all automorphisms of N. We denote by $\mathbb{I}$ the identity element of $\mathrm{Aut}(\mathsf{N})$. Let $\mathrm{Aff}(\mathsf{N})$ be the affine group of N, that is, the semidirect product $\mathrm{Aut}(\mathsf{N}) \ltimes \mathsf{N}$ under the group law

$$(A, a)(B, b) = (AB, aA(b)) \qquad \forall A, B \in \mathrm{Aut}(\mathsf{N}) \text{ and } \forall a, b \in \mathsf{N}.$$

The group $\mathrm{Aff}(\mathsf{N})$ acts on N by

$$(A, a)(y) = aA(y) \qquad \forall (A, a) \in \mathrm{Aff}(\mathsf{N}) \text{ and } \forall y \in \mathsf{N}.$$

If $(A, a) \in \mathrm{Aff}(\mathsf{N})$, let r and t be the maps that assign to (A, a) respectively the *rotational* part A and the *translational* part a.

Theorem 1.4.7 [Au] *Let* K *be a compact subgroup of* $\mathrm{Aut}(\mathsf{N})$ *and let* Γ *be a cocompact (i. e.* N/Γ *is compact), torsion free, discrete subgroup of* $\mathsf{K} \ltimes \mathsf{N}$*. Then the following statements hold:*

(a) $\Gamma \cap (\{\mathbb{I}\} \ltimes \mathsf{N})$ *is the maximal normal nilpotent subgroup of* Γ*;*

(b) the group $\Psi \stackrel{\text{def}}{=} \Gamma/(\Gamma \cap (\{\mathbb{I}\} \ltimes \mathsf{N}))$ *is finite and isomorphic to* $r(\Gamma)$*;*

(c) if Γ' *is a cocompact discrete subgroups of* $\mathsf{K} \ltimes \mathsf{N}$ *and* $\varphi : \Gamma \to \Gamma'$ *is an isomorphism, then* φ *can be uniquely extended to an automorphism* $\tilde{\varphi}$ *of* $\mathsf{K} \ltimes \mathsf{N}$ *such that* $\tilde{\varphi}(\{\mathbb{I}\} \ltimes \mathsf{N}) = \{\mathbb{I}\} \ltimes \mathsf{N}$*.*

Definition 1.4.8 Let K be a compact subgroup of $\mathrm{Aut}(\mathsf{N})$ and let Γ be a cocompact, torsion free, discrete subgroup of $\mathsf{K} \ltimes \mathsf{N}$. The quotient space $\mathcal{M}' = \mathsf{N}/\Gamma$ is a smooth riemannian closed manifold, called *infra-nil-manifold* and the finite group Ψ is the *holonomy group* of the manifold. If Ψ is the trivial group then we say that $\mathcal{M}'$ is a *nil-manifold*; in this case $\Gamma = \{\mathbb{I}\} \ltimes \mathsf{Z}$ and we will write $\mathcal{M}' = \mathsf{N}/\mathsf{Z}$.

A map $\Phi : \mathcal{M}' \to \mathcal{M}'$ is called an *endomorphism* of $\mathcal{M}'$ if it is induced by an automorphism φ of $\mathsf{K} \ltimes \mathsf{N}$ such that $\varphi(\Gamma) \subset \Gamma$.

Remark 1.4.9 By Theorem 1.4.7, we can associate to the infra-nil-manifold $\mathcal{M}' = \mathsf{N}/\Gamma$ the following exact sequence

$$0 \longrightarrow \Gamma \cap (\{\mathbb{I}\} \ltimes \mathsf{N}) \longrightarrow \Gamma \longrightarrow \Psi \longrightarrow 0. \, .$$

The action of the group Γ on N is *properly discontinuous*: for all $y \in \mathsf{N}$ there is a neighborhood V of y such that $\gamma(V) \cap V = \emptyset$ implies $\gamma = e$.

In fact, let W be a neighborhood of $(\mathbb{I}, e)$ in $\mathsf{K} \ltimes \mathsf{N}$ with compact closure. Now, let $\gamma \in \Gamma$, then

$$\gamma(yt(W)) \cap yt(W) \neq \emptyset \tag{16}$$

iff $t(\gamma(\mathbb{I}, y)W) \cap t((\mathbb{I}, y)W) \neq \emptyset$, that is, γ belongs to the set

$$(\mathbb{I}, y)W(\mathsf{K} \ltimes \{e\})W^{-1}(\mathbb{I}, y)^{-1}$$

which has compact closure. Therefore, since Γ is descrete, (16) is satisfied only for a finite number of elements γ of Γ and there is a neighborhood $V \subset yt(W)$ of y such that $\gamma(V) \cap V = \emptyset$ implies $\gamma = e$.

Since Γ acts proper discontinuously on N, then the natural projection $\pi' : \mathsf{N} \to \mathsf{N}/\Gamma = \mathcal{M}'$ is a regular covering, the universal covering $\widetilde{\mathcal{M}}'$ of $\mathcal{M}'$ is N and $\Gamma = \mathcal{D}(\widetilde{\mathcal{M}}'/\mathcal{M}') \simeq \pi_1(\mathcal{M}')$.

Example 1.4.10 For $r \geq 2$ and $1 \leq m \leq r$, $\mathrm{UT}_r^m(\mathbb{R})$ is the set of upper triangular $r \times r$ matrices with entries in $\mathbb{R}$ which have 1 on the main diagonal and 0 on the $m-1$ diagonals immediately above the main one. We will write simply $\mathrm{UT}_r(\mathbb{R})$ when $m = 1$.

It is easy to check that the commutator group $[\mathrm{UT}_r^{m_1}(\mathbb{R}), \mathrm{UT}_r^{m_2}(\mathbb{R})] = \mathrm{UT}_r^{m_3}(\mathbb{R})$ for $1 \leq m_1, m_2 \leq r$ and $m_3 \overset{\text{def}}{=} \max\{m_1 + m_2, r-1\}$. Moreover

$$\mathrm{UT}_r(\mathbb{R}) \geq \mathrm{UT}_r^2(\mathbb{R}) \geq \cdots \geq \mathrm{UT}_r^{r-1}(\mathbb{R}) \geq \{\mathbb{I}\}.$$

Therefore, $\mathrm{UT}_r(\mathbb{R})$ is a nilpotent group of class equal to $r-1$ diffeomorphic to $\mathbb{R}^{\frac{r(r-1)}{2}}$. This in fact is a typical example, because by Malcev's theorem, every connected, simply connected, nilpotent group N embeds as a closed subgroup of some $\mathrm{UT}_r(\mathbb{R})$ (see [CoGr] as a general reference).

The quotient space $\mathrm{UT}_r(\mathbb{R})/\mathrm{UT}_r(\mathbb{Z})$ where $\mathrm{UT}_r(\mathbb{Z})$ is the subgroup of integer matrices of $\mathrm{UT}_r(\mathbb{R})$, is a nil-manifold of dimension $n = \frac{r(r-1)}{2}$. As an example of infra-nil-manifold, define $\gamma = (U, u) \in \mathrm{Aff}(\mathrm{UT}_3(\mathbb{R}))$ by

$$U(\begin{bmatrix} 1 & y_1 & y_3 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{bmatrix}) = \begin{bmatrix} 1 & y_1 & -y_3 \\ 0 & 1 & -y_2 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let Γ be the group generated by $\{\mathbb{I}\} \ltimes \mathrm{UT}_3(\mathbb{Z})$ and γ then $\mathrm{UT}_3(\mathbb{R})/\Gamma$ is an infra-nil-manifold with $\Psi \simeq \mathbb{Z}_2$.

Remark 1.4.11 Note that an endomorphism Φ of an infra-nil-manifold $\mathcal{M}'$ belongs to $\mathcal{E}^1(\mathcal{M}')$ iff the differential of $\widetilde{\Phi}$ at the identity element of N has all the eigenvalues greater than one in absolute value.

The following two results are fundamental for the proof of the next theorem.

Theorem 1.4.12 *If Γ is a finitely generated group then:*

(a) [Gr] *if Γ has polynomial growth, Γ contains a nilpotent subgroup of finite index.*

(b) [AuSc] *if Γ contains a nilpotent subgroup of finite index and it is torsion free, then Γ is isomorphic to a cocompact, torsion free, discrete subgroup of $\mathsf{K} \ltimes \mathsf{N}$, where N is a connected, simply connected nilpotent Lie group and K a finite subgroup of* $\mathrm{Aut}(\mathsf{N})$.

Here is the Conjugation Theorem.

Theorem 1.4.13 [Hira1] *If $F \in \mathcal{E}(\mathcal{M})$, there exist an infra-nil-manifold $\mathcal{M}'$ and an endomorphism $\Phi \in \mathcal{E}^1(\mathcal{M}')$ such that F and Φ are topologically conjugate:*

$$\alpha_0 \circ F = \Phi \circ \alpha_0,$$

where $\alpha_0 : \mathcal{M} \to \mathcal{M}'$ is an homeomorphism.

Proof. By Theorem 1.4.3, the finitely generated group $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ has polynomial growth and, by (a) of Theorem 1.4.12, this implies that $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ contains a nilpotent subgroup of finite index.

Since by (b) of Theorem 1.4.1 $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ is torsion free, by (b) of Theorem 1.4.12, the group $\mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ is isomorphic to a cocompact, torsion free, discrete subgroup Γ of $\mathsf{K}\ltimes\mathsf{N}$ where N is a connected, simply connected nilpotent Lie group and K a finite subgroup of $\mathrm{Aut}(\mathsf{N})$. Let $\mathcal{M}'$ be the infra-nil-manifold N/Γ and let $\psi : \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M}) \to \Gamma$ be the isomorphism. The homomorphism $\widetilde{F}^\sharp : \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M}) \to \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ is injective because if $\widetilde{F}^\sharp(\gamma) = e$, then

$$\widetilde{F}(\gamma(y)) = \widetilde{F}^\sharp(\gamma)(\widetilde{F}(y)) = \widetilde{F}(y) \qquad \forall\gamma \in \Gamma \text{ and } \forall y \in \widetilde{\mathcal{M}},$$

and, since, by Theorem 1.4.7, $\widetilde{F}$ is a homeomorphism, this means that $\gamma = e$. Therefore, by (c) of Theorem 1.4.7, the injective homomorphism $\psi \circ \widetilde{F}^\sharp \circ \psi^{-1}$ of Γ, can be extended uniquely to an automorphism $\varphi : \mathsf{K}\ltimes\mathsf{N} \to \mathsf{K}\ltimes\mathsf{N}$ such that $\varphi(\{\mathbb{I}\}\ltimes\mathsf{N}) = \{\mathbb{I}\}\ltimes\mathsf{N}$. Therefore, the automorphism φ induces an endomorphism $\Phi : \mathcal{M}' \to \mathcal{M}'$. Let $\widetilde{\Phi}$ be the lifting of Φ defined by the restriction of φ to $\{\mathbb{I}\}\ltimes\mathsf{N}$, then $\widetilde{\Phi}^\sharp = \psi \circ \widetilde{F}^\sharp \circ \psi^{-1}$.

By Remark 1.3.7, since F is expanding, there exists a continuous surjective map $\alpha_1 : \mathcal{M}' \to \mathcal{M}$ such that $F \circ \alpha_1 = \alpha_1 \circ \Phi$.

Let $l(\mathsf{N})$ be the Lie algebra of N and let $\exp : l(\mathsf{N}) \to \mathsf{N}$ be the exponential map. Then exp is an homeomorphism and $\widetilde{\Phi} \circ \exp = \exp \circ D_e\widetilde{\Phi}$ where $D_e\widetilde{\Phi} : l(\mathsf{N}) \to l(\mathsf{N})$ is the differential of $\widetilde{\Phi}$ at the identity of N. Thus

$$\widetilde{F}^k \circ (\tilde{\alpha}_1 \circ \exp) = (\tilde{\alpha}_1 \circ \exp) \circ (D_e\widetilde{\Phi})^k \qquad \forall k \geq 1. \tag{17}$$

Now, we show that all the eigenvalues of $D_e\widetilde{\Phi}$ are greater than one in modulus, i. e. $\Phi \in \mathcal{E}^1(\mathcal{M}')$. Suppose, on the contrary, that $D_e\widetilde{\Phi}$ has an eigenvalue λ with $|\lambda| \leq 1$. Then, if $z \in l(\mathsf{N})$ is a corresponding eigenvector with $||z|| = 1$, $(D_e\widetilde{\Phi})^k z = \lambda^k z$ for all $k \geq 1$. On the other hand, since $\widetilde{F}$ is expanding, the sequence $\{\widetilde{F}^k(y)\}_{k\geq 1}$ is unbounded for all $y \in \widetilde{\mathcal{M}}$ except the unique fixed point y_0 of $\widetilde{F}$. Hence, by (17), the sequence

$$\{\widetilde{F}^k((\tilde{\alpha}_1 \circ \exp)(z))\}_{k\geq 1} \text{ is contained in } (\tilde{\alpha}_1 \circ \exp)(\{tz \ : \ |t| \leq 1\})$$

which is compact. Hence $(\tilde{\alpha}_1 \circ \exp)(z) = y_0$ and the same argument shows that $(\tilde{\alpha}_1 \circ \exp)(rz) = y_0$ for all $r \in \mathbb{R}$.

To prove that this is a contradiction, it suffices to prove that $\tilde{\alpha}_1$, and hence also $\tilde{\alpha}_1 \circ \exp$, is a proper map. In fact, let C' be a covering domain of $\mathcal{M}'$ then, since α_1 is surjective, $\pi(\tilde{\alpha}_1(C')) = \mathcal{M}$. If $K \subset \widetilde{\mathcal{M}}$ is compact, the set

$$\Gamma_1 \stackrel{\text{def}}{=} \{\gamma \in \Gamma \,:\, \gamma(\tilde{\alpha}_1(C')) \cap K \neq \emptyset\}$$

is finite and $K \subset \cup_{\gamma \in \Gamma_1} \gamma(\tilde{\alpha}_1(C'))$. Thus,

$$\tilde{\alpha}_1^{-1}(K) \subset \bigcup_{\gamma \in \Gamma_1} (\tilde{\alpha}_1^{\sharp})^{-1}(\gamma)(C')$$

because $\gamma(\tilde{\alpha}_1(C')) = \tilde{\alpha}_1((\tilde{\alpha}_1^{\sharp})^{-1}(\gamma)(\overline{C'}))$. That is, the closed set $\tilde{\alpha}_1^{-1}(K)$ is contained in a finite union of compact sets and therefore it is compact.

Now, that we have established that the endomorphism Φ is expanding on $\mathcal{M}'$, we can apply Theorem 1.3.6 and find the homeomorphism α_0 which conjugate F to Φ. Q.E.D.

Remark 1.4.14 In general, little can be said about the regularity of the homeomorphism α_0. However, C. Arteaga has proved in [Ar] that if $F \in \mathcal{E}^1(\mathcal{M})$ and $\Phi \in \mathcal{E}^1(\mathcal{M})$ is the associated endomorphism, then the homeomorphism $\alpha_0 : \mathcal{M} \to \mathcal{M}'$, which realizes the conjugation, is absolutely continuous iff

$$|\det(D_x(F^k))| = |\det(D_{\alpha_0(x)}(\Phi^k))| \ \forall k \geq 1 \text{ and } \forall x \in \mathcal{M} \text{ such that } F^k(x) = x.$$

Moreover, if F, Φ are C^r-maps with $r \geq 2$, then α_0 is a C^{r-1}-diffeomorphism.

§1.5 A comparison with the one-sided shift.

Let Σ_N be the set of sequences with values in $\{0, \ldots, N-1\}$ with the product discrete topology. The *one-sided shift* $S : \Sigma_N \to \Sigma_N$ is the map defined by

$$S(\sigma) \stackrel{\text{def}}{=} \{i_k\}_{k \geq 2} \qquad \forall \sigma = \{i_k\}_{k \geq 1} \in \Sigma_N.$$

Theorem 1.5.1 [Hi] *If $F \in \mathcal{E}(\mathcal{M})$ with degree N, there is a continuous surjective map $h : \Sigma_N \to \mathcal{M}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \Sigma_N & \xrightarrow{S} & \Sigma_N \\ h\downarrow & & \downarrow h \\ \mathcal{M} & \xrightarrow{F} & \mathcal{M} \end{array}.$$

Proof. Let $\Gamma = \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ and choose a lifting $\widetilde{F}$ of F. Define $\mathcal{A} \stackrel{\text{def}}{=} \{\gamma_0, \ldots, \gamma_{N-1}\}$ as the set of right coset representatives for $\Gamma/\widetilde{F}^\sharp(\Gamma)$ with $\gamma_0 = e$. Note that, if $x \in \mathcal{M}$ and $y \in \pi^{-1}(x)$, then

$$F^{-1}(x) = \{\pi((\widetilde{F}^{-1} \circ \gamma)(y)) \ : \ \gamma \in \mathcal{A}\}. \tag{18}$$

Let $y_0 \in \widetilde{\mathcal{M}}$ be the unique fixed point of $\widetilde{F}$. If $\sigma = \{i_k\}_{k\geq 1}$ is a sequence in Σ_N, define $h(\sigma) \in \mathcal{M}$ by $h(\sigma) = (\pi \circ \tilde{h})(\sigma)$, where

$$\tilde{h}(\sigma) \stackrel{\text{def}}{=} \lim_{k\to\infty} (\widetilde{F}^{-1} \circ \gamma_{i_1} \circ \ldots \circ \widetilde{F}^{-1} \circ \gamma_{i_k})(y_0).$$

If λ denotes the expanding constant of F, the limit exists because, by Theorem 1.3.3, $\widetilde{F}^{-1}$ is a $\frac{1}{\lambda}$-contraction and Γ is a set of isometries with respect to the metric $\tilde{d}$ on $\widetilde{\mathcal{M}}$. Therefore

$$\tilde{d}((\widetilde{F}^{-1} \circ \gamma_{i_1} \circ \ldots \circ \widetilde{F}^{-1} \circ \gamma_{i_k})(y_0), (\widetilde{F}^{-1} \circ \gamma_{i_1} \circ \ldots \circ \widetilde{F}^{-1} \circ \gamma_{i_{k+1}})(y_0)) \leq \frac{R}{\lambda^k}$$

with $R \stackrel{\text{def}}{=} \max\{\tilde{d}(y_0, (\widetilde{F}^{-1} \circ \gamma)(y_0)) \ : \ \gamma \in \mathcal{A}\}$. This also proves that

$$\tilde{d}(y_0, \tilde{h}(\sigma)) \leq R \sum_{k=0}^{\infty} \frac{1}{\lambda^k} = \frac{R}{\lambda - 1}.$$

The map $\tilde{h}$ is continuous and therefore also h: if $\sigma, \sigma' \in \Sigma_N$ have the first k elements equal, then

$$\tilde{d}(\tilde{h}(\sigma), \tilde{h}(\sigma')) \leq \frac{1}{\lambda^k} \tilde{d}(\tilde{h}(S^k(\sigma)), \tilde{h}(S^k(\sigma')) \leq \frac{1}{\lambda^k} \frac{2R}{\lambda - 1}.$$

The map $\tilde{h}$ is surjective because the compact set $h(\Sigma_N)$ contains, by (18)

$$\bigcup_{k_0\geq 1} \{h(\sigma) \ : \ i_k = 0 \ \ \forall k \geq k_0\} = \bigcup_{k\geq 0} F^{-k}(\pi(y_0)),$$

which is dense in $\mathcal{M}$, by (b) of Proposition 1.3.4.

Finally, we check that the equality $F \circ h = h \circ S$ holds: if $\sigma \in \Sigma_N$ then

$$(F \circ h)(\sigma) = (F \circ \pi \circ \tilde{h})(\sigma) = (\pi \circ \widetilde{F} \circ \tilde{h})(\sigma) = (\pi \circ \gamma_{i_1} \circ \tilde{h} \circ S)(\sigma) = (h \circ S)(\sigma)$$

because $\pi \circ \gamma = \pi$ for all $\gamma \in \Gamma$. Q.E.D.

Remark 1.5.2 Note that, the map h is not injective in general. In fact, if Φ is an expanding endomorphism on $\mathbf{T}^1$ such that $\widetilde{\Phi}$ is the multiplication by $N \geq 2$ on $\mathbb{R}$ then we can take $\mathcal{A}$ as the set of traslations $\gamma_i(x) = x + i$ for $i = 0, \ldots, N-1$.

Therefore, by the definition of the map h,

$$h(\sigma) = \pi(\sum_{k=1}^{\infty} \frac{i_k}{N^k}) \quad \forall \sigma = \{i_k\}_{k \geq 1} \in \Sigma_N.$$

Let σ and σ' be the constant sequences respectively equal to 0 and $N-1$. Then $\sigma \neq \sigma'$ and

$$h(\sigma') = \pi(\sum_{k=1}^{\infty} \frac{N-1}{N^k}) = \pi(1) = \pi(\gamma_1(0)) = \pi(0) = h(\sigma).$$

2

Ergodic properties

In this Chapter, we provide a collection of ergodic properties of an expanding map F on a closed manifold $\mathcal{M}$. First we discuss a scheme for searching invariant measures of F due essentially to P. Walters ([Wa1]) and here adapted to the case of the expanding maps. This scheme is based on the *transfer operator* $\mathcal{L} : C(\mathcal{M}, \mathbb{R}) \to C(\mathcal{M}, \mathbb{R})$ defined as

$$\mathcal{L}(f)(x) \stackrel{\text{def}}{=} \sum_{y \in F^{-1}(x)} e^{\varphi(y)} f(y) \qquad \forall x \in \mathcal{M}$$

for a suitable choice of $\varphi \in C(\mathcal{M}, \mathbb{R})$. More precisely, if

$$C(x, x') \stackrel{\text{def}}{=} \sup_{k \geq 1} \sup_{y \in F^{-k}(x)} \{\sum_{i=0}^{k-1} |\varphi(F^i(y)) - \varphi(F^i(y'))|\} \leq L d(x, x')$$

for some constant $L \geq 0$, when $d(x, x') \leq \epsilon_1$, then, the set

$$\Lambda \stackrel{\text{def}}{=} \{f \in C(\mathcal{M}, \mathbb{R}) \ : \ f \geq 0 \text{ and } f(x) \leq e^{C(x,x')} f(x') \text{ if } d(x, x') \leq \epsilon_1\}$$

is a non-empty closed convex set of $C(\mathcal{M}, \mathbb{R})$. and $\mathcal{L}(\Lambda) \subset \Lambda$. When it is known a fixed point h of $\mathcal{L}$ in Λ such that $h > 0$, then there exists a F-invariant normalized measure μ. Moreover, F is *exact* with respect to μ: every set belonging to the intersection

$$\bigcap_{k=0}^{\infty} F^{-k}(\text{Bor}(\mathcal{M}))$$

has measure equal to either 0 or 1 (exactness is stronger then ergodicity).

Two remarkable invariant measures, which originally had been found by different methods, now come out by the above scheme choosing opportunely the map φ. The first invariant measure, written μ_0, corresponds to $\varphi(x) = -\log(N)$, where $N \geq 2$ is the degree of F, and is the measure with maximal entropy. The second one, written μ_F, holds when F is a C^2-differentiably expanding map on a smooth riemannian closed manifold. μ_F corresponds to $\varphi(x) = -\log(|\det(D_x F)|)$ and it is equivalent to the Lebesgue measure. In view of some new applications of these two measures, some preparatory results end this Chapter.

§2.1 Transfer operator and invariant measures.

Let $\mathcal{M}$ be a closed manifold and let $F \in \mathcal{E}(\mathcal{M})$.

Definition 2.1.1 We define the *transfer operator* $\mathcal{L} : C(\mathcal{M}, \mathbb{R}) \to C(\mathcal{M}, \mathbb{R})$ associated to F and $\varphi \in C(\mathcal{M}, \mathbb{R})$ as

$$\mathcal{L}(f)(x) \stackrel{\text{def}}{=} \sum_{y \in F^{-1}(x)} e^{\varphi(y)} f(y) \qquad \forall x \in \mathcal{M}$$

where $f \in C(\mathcal{M}, \mathbb{R})$.

Remark 2.1.2 Note that, since F is a self-covering with degree $N \geq 2$ the sum in the definition of $\mathcal{L}$ is finite. Moreover, if $d(x, x') \leq \epsilon_1$, by Theorem 1.2.7, letting $D \stackrel{\text{def}}{=} \{x, x'\}$, there is a natural bijection between the sets $\bigcup_{k=1}^{\infty} F^{-k}(x)$ and $\bigcup_{k=1}^{\infty} F^{-k}(x')$ by letting, for $k \geq 1$, $y \in F^{-k}(x)$ correspond to $y' \in F^{-k}(x')$ iff y and y' lie in the same component $D_i^{(k)}$; moreover

$$\epsilon_1 \geq d(x, x') = d(F^k(y), F^k(y')) \geq \lambda^k d(y, y'). \tag{1}$$

Lemma 2.1.3 *Let $\varphi \in C(\mathcal{M}, \mathbb{R})$ be such that, for some constant $L \geq 0$*

$$C(x, x') \stackrel{\text{def}}{=} \sup_{k \geq 1} \sup_{y \in F^{-k}(x)} \{\sum_{i=0}^{k-1} |\varphi(F^i(y)) - \varphi(F^i(y'))|\} \leq L d(x, x')$$

when $d(x, x') \leq \epsilon_1$. Then, the set

$$\Lambda \stackrel{\text{def}}{=} \{f \in C(\mathcal{M}, \mathbb{R}) \ : \ f \geq 0 \text{ and } f(x) \leq e^{C(x,x')} f(x') \text{ if } d(x, x') \leq \epsilon_1\}$$

is a non-empty closed convex set of $C(\mathcal{M}, \mathbb{R})$ and $\mathcal{L}(\Lambda) \subset \Lambda$.

Proof. Since $C(x,x') \geq 0$ and does not depend on f, the non-negative constant maps belong to Λ and Λ is closed and convex.

Let $x, x' \in \mathcal{M}$ be such that $d(x,x') \leq \epsilon_1$. By (1), $d(y,y') \leq \epsilon_1$. If $f \in \Lambda$ then $f(y) \leq e^{C(y,y')} f(y')$ and

$$\mathcal{L}(f)(x) \leq \sum_{y' \in F^{-1}(x')} e^{\varphi(y')} f(y') e^{|\varphi(y)-\varphi(y')|+C(y,y')} \leq \mathcal{L}(f)(x') e^{C(x,x')}$$

because $|\varphi(y) - \varphi(y')| + C(y,y') \leq C(x,x')$. Therefore $\mathcal{L}(f) \in \Lambda$. Q.E.D.

We denote by $\mathrm{Meas}(\mathcal{M})$ the space of all positive measures on $\mathcal{M}$ with respect to the σ-algebra of the borelians $\mathrm{Bor}(\mathcal{M})$ equipped with the weak*-topology. We indicate by $\mathrm{Meas}_1(\mathcal{M})$ the set of all normalized measures. If $\mu \in \mathrm{Meas}(\mathcal{M})$ we define $\mu(f) \stackrel{\text{def}}{=} \int_{\mathcal{M}} f(x) d\mu(x)$ for all $f \in L^1_\mu(\mathcal{M}, \mathbb{R})$.

Definition 2.1.4 If F is a measurable map of $\mathcal{M}$, then a measure $\mu \in \mathrm{Meas}_1(\mathcal{M})$ is F-*invariant* if $\mu(F^{-1}(E)) = \mu(E)$ for all $E \in \mathrm{Bor}(\mathcal{M})$. In this case, F is *exact* with respect to μ when the condition $F^{-k}(F^k(E)) = E \in \mathrm{Bor}(\mathcal{M})$ holding for all $k \geq 1$ implies that $\mu(E)$ is equal to either 0 or 1. This property can be expressed also in the following equivalent way: every set belonging to the intersection

$$\bigcap_{k=0}^{\infty} F^{-k}(\mathrm{Bor}(\mathcal{M}))$$

has measure equal to either 0 or 1 (see [Ro] as a general reference).

Remark 2.1.5 The exactness property of a measurable map F of $\mathcal{M}$ implies the following facts:

(a) if $F^k(E) \in \mathrm{Bor}(\mathcal{M})$ for all $k \geq 0$ and $\mu(E) > 0$ then

$$\lim_{k\to\infty} \mu(F^k(E)) = 1.$$

In fact, let $E_k = F^{-k}(F^k(E))$ for all $k \geq 0$. Then $\{E_k\}_{k\geq 0}$ is an increasing sequence of sets. If $S \stackrel{\text{def}}{=} \bigcup_{i=0}^{\infty} E_i$, then, for $k \geq 0$,

$$S = \bigcup_{i=k}^{\infty} E_i = F^{-k}(F^k(S)) \;\text{ and }\; \mu(S) \geq \mu(E) > 0.$$

Therefore, by the exactness property

$$\lim_{k\to+\infty}\mu(F^k(E))=\lim_{k\to+\infty}\mu(E_k)=\mu(S)=1.$$

(b) F is *strong mixing*[1]: if $E,E'\in \mathrm{Bor}(\mathcal{M})$ then

$$\lim_{k\to\infty}\mu(F^{-k}(E)\cap E')=\mu(E)\mu(E').$$

In fact, let U_F be the operator defined by $U_F(f)=f\circ F$ for all $f\in L^2_\mu(\mathcal{M},\mathbb{R})$. Once, we prove that

$$\lim_{k\to\infty} < U_F^k(f),g>_{L^2}=< f,1>_{L^2}< 1,g>_{L^2} \qquad \forall f,g\in L^2_\mu(\mathcal{M},\mathbb{R}) \tag{2}$$

the strong mixing property will follow by choosing $f=I_E$ and $g=I_{E'}$.

Note that

$$< U_F^k(f),g>_{L^2}=< U_F^k(f-< f,1>_{L^2}),g>_{L^2}+< f,1>_{L^2}< 1,g>_{L^2}\ . \tag{3}$$

Rohlin has proved in [Ro] that F is exact with respect to μ iff the intersection subspace

$$\bigcap_{k=0}^{\infty} U_F^k(L^2_\mu(\mathcal{M},\mathbb{R}))$$

contains only the costant maps. Then $||U_F^k(f-< f,1>)||_{L^2}\to 0$ as $k\to\infty$ and, by (3), the equation (2) holds.

Let $\varphi\in C(\mathcal{M},\mathbb{R})$ be such that it satisfies the hypothesis of Lemma 2.1.3. Then the following theorem yields a procedure to find an F-invariant measure μ for F when a fixed point h belonging to set Λ of the transfer operator $\mathcal{L}$ is known.

Theorem 2.1.6 [Wa1] *Assume that there is a map $h\in\Lambda$ such that $\mathcal{L}(h)=h$ and $h>0$, and define the operator $\widetilde{\mathcal{L}}$ as*

$$\widetilde{\mathcal{L}}(f)\stackrel{\text{def}}{=}\mathcal{L}\left(\frac{hf}{h\circ F}\right) \qquad \forall f\in C(\mathcal{M},\mathbb{R}).$$

[1] In particular F is *ergodic*: if $E\in\mathrm{Bor}(\mathcal{M})$ such that $F^{-1}(E)=E$ then $\mu(E)$ is equal to either 0 or 1 (it suffices to choose in the definition of the strong mixing property $E'=E$).

Then there exists $\mu \in \mathrm{Meas}_1(\mathcal{M})$ *such that:*

(a) μ *is an* F*-invariant and*

$$\lim_{k\to\infty} \widetilde{\mathcal{L}}^k(f) = \mu(f) \qquad \forall f \in C(\mathcal{M}, \mathbb{R});$$

(b) F *is exact with respect to* μ.

Proof. First, note that, since $\mathcal{L}(h) = h$ and $h > 0$ then

$$||\widetilde{\mathcal{L}}(f)|| \le \sum_{y\in F^{-1}(x)} \frac{e^{\varphi(y)}h(y)}{h(x)}|f(y)| \le \frac{\mathcal{L}(h)(x)}{h(x)}||f|| = ||f|| \quad \forall f \in C(\mathcal{M}, \mathbb{R}).$$

Since $\widetilde{\mathcal{L}}(1) = 1$, then $||\widetilde{\mathcal{L}}|| = 1$.

By Theorem 1.2.7, if $x, x' \in \mathcal{M}$ with $d(x, x') \le \epsilon_1$ and $k \ge 1$, then, since $\mathcal{L}(h) = h$,

$$\begin{aligned}
|\widetilde{\mathcal{L}}^k(f)(x) - \widetilde{\mathcal{L}}^k(f)(x')| \;\le\; & \sum_{y\in F^{-k}(x)} \frac{e^{\varphi_k(y)}h(y)}{h(x)}|f(y) - f(y')| \\
& + \sum_{y\in F^{-k}(x)} |\frac{e^{\varphi_k(y)}h(y)}{h(x)} - \frac{e^{\varphi_k(y')}h(y')}{h(x')}||f(y')| \\
\le\; & \sup\{|f(z) - f(z')| : z, z' \in \mathcal{M},\ d(z, z') \le \frac{\epsilon_1}{\lambda^k}\} \\
& + ||f|| \sup_{y\in F^{-k}(x)} \{|1 - e^{|\varphi_k(y)-\varphi_k(y')|}\frac{h(y')}{h(y)}\frac{h(x)}{h(x')}|\}
\end{aligned}$$

where $\varphi_k(y) \overset{\text{def}}{=} \sum_{i=0}^{k-1} \varphi(F^k(y))$ for all $y \in \mathcal{M}$.

Since $h \in \Lambda$ and $|\varphi_k(y) - \varphi_k(y')| + C(y, y') \le C(x, x')$, we have

$$e^{-2Ld(x,x')} \le e^{-2C(x,x')} \le e^{\varphi_k(y)-\varphi_k(y')}\frac{h(y')}{h(y)}\frac{h(x)}{h(x')} \le e^{2C(x,x')} \le e^{2Ld(x,x')}.$$

Therefore, the set $\{\widetilde{\mathcal{L}}^k(f) : k \ge 1\}$ is equicontinuous and bounded in $C(\mathcal{M}, \mathbb{R})$, and, since $\mathcal{M}$ is compact, by Ascoli's theorem, there exists a converging subsequence $\widetilde{\mathcal{L}}^{k_i}(f) \to m(f) \in C(\mathcal{M}, \mathbb{R})$.

Now we prove that, actually, $m(f)$ is a constant map. For $k \geq 1$, since $\widetilde{\mathcal{L}}^k(f)(x)$ is a strict convex combination of the real numbers $f(y)$ with $y \in F^{-k}(x)$, then

$$\min_{\mathcal{M}}\{f\} \leq \min_{\mathcal{M}}\{\widetilde{\mathcal{L}}^k(f)\} \leq \min_{\mathcal{M}}\{m(f)\} = \min_{\mathcal{M}}\{\widetilde{\mathcal{L}}^k(m(f))\}. \tag{4}$$

Moreover, if $z \in \mathcal{M}$ is such that

$$\widetilde{\mathcal{L}}^k(m(f))(z) = \min_{\mathcal{M}}\{\widetilde{\mathcal{L}}^k(m(f))\} = \min_{\mathcal{M}}\{m(f)\},$$

then, for all $x \in F^{-k}(z)$, $m(f)(x) = \min_{\mathcal{M}}\{m(f)\}$. Therefore, since by (b) of Proposition 1.3.4, $\cup_{k\geq 1}F^{-k}(z)$ is dense in $\mathcal{M}$, $m(f) \equiv \min_{\mathcal{M}}\{m(f)\}$, that is, $m(f)$ is a constant map. By (4), we deduce that the entire sequence $\{\widetilde{\mathcal{L}}^k(f)\}_{k\geq 1}$ converges to the costant map $m(f)$.

Thus we have a bounded linear map $m : C(\mathcal{M}, \mathbb{R}) \to \mathbb{R}$, and by Riesz representation theorem, since $m(1) = \lim_{k\to\infty} \widetilde{\mathcal{L}}^k(1) = 1$, then $m \in \mathrm{Meas}_1(\mathcal{M})$. So, we can write μ instead of m.

(a) If $f \in C(\mathcal{M}, \mathbb{R})$ and $x \in \mathcal{M}$ then

$$\widetilde{\mathcal{L}}(f \circ F)(x) = \sum_{y\in F^{-1}(x)} \frac{e^{\varphi(y)}h(y)}{h(x)} f(F(y)) = f(x)\frac{\mathcal{L}(h)(x)}{h(x)} = f(x).$$

Therefore, $\mu(f \circ F) = \lim_{k\to\infty} \widetilde{\mathcal{L}}^k(f \circ F) = \lim_{k\to\infty} \widetilde{\mathcal{L}}^{k-1}(f) = \mu(f)$, that is, μ is F-invariant.

(b) Since the sequence $\{\widetilde{\mathcal{L}}^k(f)\}_{k\geq 1}$ converges uniformly to $\mu(f)$ for all $f \in C(\mathcal{M}, \mathbb{R})$, and $C(\mathcal{M}, \mathbb{R})$ is dense in $L^1_\mu(\mathcal{M}, \mathbb{R})$, then $\{\widetilde{\mathcal{L}}^k(f)\}_{k\geq 1}$ L^1_μ-converges to $\mu(f)$ for all $f \in L^1_\mu(\mathcal{M}, \mathbb{R})$. If $E \in \mathrm{Bor}(\mathcal{M})$ is such that $F^{-k}(F^k(E)) = E$ for all $k \geq 1$, then it is simply to verify that

$$\widetilde{\mathcal{L}}^k(I_E f)(F^k(x)) = I_E(x)\widetilde{\mathcal{L}}^k(f)(F^k(x)) \quad \forall x \in \mathcal{M} \text{ and } \forall k \geq 1. \tag{5}$$

Therefore, by (5), since $\widetilde{\mathcal{L}}(1) = 1$,

$$\mu(E) = \mu(I_E(\widetilde{\mathcal{L}}^k(1) \circ F^k)) = \mu(\widetilde{\mathcal{L}}^k(I_E) \circ F^k) = \mu(I_E(\widetilde{\mathcal{L}}^k(I_E) \circ F^k)),$$

and taking the limit as $k \to \infty$, we obtain that $\mu(E) = \mu(I_E \mu(I_E)) = (\mu(E))^2$ that is, $\mu(E)$ is equal to either 0 or 1. Hence, F is exact with respect to μ. Q.E.D.

§2.2 Two remarkable invariant measures.

By the scheme described in Theorem 2.1.6, we will find two invariant measures. Here is the first one which it has been studied by M. Misiurewicz in [Mi].

Theorem 2.2.1 [Mi] *Let $F \in \mathcal{E}(\mathcal{M})$ and define $\varphi(x) \stackrel{\text{def}}{=} -\log(N)$ where $N \geq 2$ is the degree of F. Then*

$$\lim_{k\to\infty} \mathcal{L}^k(f) = \mu_0(f) \qquad \forall f \in C(\mathcal{M}, \mathbb{R}),$$

where μ_0 is a normalized F-invariant measure for which F is exact. Moreover, the measure μ_0 vanishes on points and it is positive on non-empty open sets.

Proof. The function $C(x, x')$ defined in Lemma 2.1.3 is identically 0 and the set Λ contains only non-negative constant maps. Since

$$\mathcal{L}(1)(x) = \frac{1}{N} \sum_{y \in F^{-1}(x)} 1 = 1$$

we can choose $h \equiv 1$ and by Theorem 2.1.6 we find the wanted measure μ_0.

If $E \in \mathrm{Bor}(\mathcal{M})$ with $\mu_0(\partial E) = 0$, since $\mathcal{L}^k(f) = \widetilde{\mathcal{L}}^k(f) \to \mu_0(f)$, as $k \to \infty$, for all $f \in C(\mathcal{M}, \mathbb{R})$, then, for any $a \in \mathcal{M}$

$$\mu_0(E) = \lim_{k\to\infty} \frac{1}{N^k} \mathrm{card}(E \cap F^{-k}(a)). \tag{6}$$

For $x \in \mathcal{M}$, (6) yields

$$1 \geq \mu_0(\{F^k(x)\}) = N^k \mu_0(\{x\}) \qquad \forall k \geq 0,$$

hence $\mu_0(\{x\}) = 0$.

For any non-empty open subset U of $\mathcal{M}$, there exists a non-empty open set $V \subset U$ such that $\mu_0(\partial V) = 0$. Moreover, by (a) of Proposition 1.3.4, there is a k_0 such that $F^{k_0}(V) = \mathcal{M}$. If $a \in \mathcal{M}$, there are at least N^k counterimages of a under F^{k+k_0}, belonging to V. Therefore by (6)

$$\mu_0(U) \geq \mu_0(V) \geq \lim_{k\to\infty} \frac{N^k}{N^{k+k_0}} = \frac{1}{N^{k_0}} > 0.$$

Q.E.D.

Remark 2.2.2 It has been proved by T. N. T. Goodman (see for example [Wa2] p.187) that, for a continuous map F of a compact metric space X, the following equality holds:

$$h_{\text{top}}(F) = \sup\{h_\mu(F) \ : \ \mu \in \text{Meas}_1(X) \text{ and it is } F\text{-invariant}\}$$

where $h_{\text{top}}(F)$ is the *topological entropy*, and $h_\mu(F)$ is *entropy* with respect to the measure μ of the map F.

In our case, when F is expanding, M. Misiurewicz in [Mi] has proved that μ_0 is the unique normalized measure invariant for F with *maximal entropy*, i. e. μ_0 realizes the sup:

$$h_{\mu_0}(F) = h_{\text{top}}(F) = \log(N) \geq \log(2) > 0.$$

Note that the measure μ_0 corresponds to the uniform invariant ergodic measure of the one-sided shift associated to the expanding map F in the Theorem 1.5.1.

The following theorem yields the second invariant measure, which was investigated by K. Krzyzewski and W. Szlenk in [KrSz]. Let $\mathcal{M}$ be a smooth riemannian closed manifold and indicate by ν the normalized Lebesgue measure induced on $\mathcal{M}$ by the riemannian metric.

Remark 2.2.3 Note that if $\varphi \in C^2(\mathcal{M}, \mathbb{R})$, then the associated map $C(x, x')$ satisfies the condition requested in Lemma 2.1.3, that is, there exists a costant $L \geq 0$ such that $C(x, x') \leq Ld(x, x')$, when $d(x, x') \leq \epsilon_1$.

In fact, let $y \in F^{-k}(x)$ then, since $\varphi \in C^2(\mathcal{M})$, $L' \stackrel{\text{def}}{=} \sup_{z \in \mathcal{M}}\{|\varphi'(z)|\}$ is finite. Moreover, by (1), $d(x, x') \geq \lambda^k d(F^i(y), F^i(y'))$ because $F^i(y) \in F^{-(k-i)}(x)$ and therefore the following inequalities hold

$$\sum_{i=0}^{k-1} |\varphi(F^i(y)) - \varphi(F^i(y'))| \leq L' \sum_{i=0}^{k-1} d(F^i(y), F^i(y')) \leq \frac{L'}{\lambda - 1} d(x, x').$$

We choose $L \stackrel{\text{def}}{=} \frac{L'}{\lambda-1}$.

A measure μ is *equivalent* to ν, written $\mu \sim \nu$, if, for all $E \in \text{Bor}(\mathcal{M})$, $\mu(E) = 0$ iff $\nu(E) = 0$.

Theorem 2.2.4 [KrSz] *Let $F \in \mathcal{E}^1(\mathcal{M}) \cap C^2(\mathcal{M}, \mathcal{M})$ and define $\varphi(x) \stackrel{\text{def}}{=} -\log(|\det(D_x F)|)$ then there exists a normalized F-invariant measure $\mu_F \sim \nu$ for which F is exact. Moreover*

(a) $\mu_F = \nu$ iff $\mathcal{L}(1) = 1$;

(b) the sequence of measures $\{\nu \circ F^{-k}\}_{k \geq 0}$ converges weakly to μ_F.

Proof. First we prove that $\mathcal{L}^*(\nu) = \nu$. If $\text{diam}(D) \leq \epsilon_1$, by Theorem 1.2.7, $F^{-1}(D) = D_1 \cup \ldots \cup D_N$. Since $F_i \stackrel{\text{def}}{=} F_{|D_i}$ for $i = 1, \ldots, N$ are all diffeomorphisms, for all $y \in D_i$

$$\frac{d(\nu \circ F_i)}{d\nu}(y) = |\det(D_y F)| = e^{-\varphi(y)} \quad \text{and} \quad \frac{d(\nu \circ F_i^{-1})}{d\nu}(F_i(y)) \cdot \frac{d(\nu \circ F_i)}{d\nu}(y) = 1.$$

Now, if $D \in \text{Bor}(\mathcal{M})$, for all $f \in C(\mathcal{M}, \mathbb{R})$,

$$\begin{aligned} \int_D \mathcal{L}(f)(x) d\nu(x) &= \int_D \sum_{y \in F^{-1}(x)} \frac{f(y)}{|\det(D_y F)|} d\nu(x) \\ &= \int_D \sum_{i=1}^{N} f(F_i^{-1}(x)) \frac{d(\nu \circ F_i^{-1})}{d\nu}(x) d\nu(x) \end{aligned}$$

$$\begin{aligned} &= \int_D \sum_{i=1}^{N} f(F_i^{-1}(x))d\nu(F_i^{-1}(x)) \\ &= \sum_{i=1}^{N} \int_{F_i^{-1}(D)} f(y)d\nu(y) = \int_{F^{-1}(D)} f(y)d\nu(y). \end{aligned} \tag{7}$$

Let $\{D^1, \ldots, D^m\} \subset \mathrm{Bor}(\mathcal{M})$ be a partition of $\mathcal{M}$ such that $\mathrm{diam}(D^j) \leq \epsilon_1$ for each $j = 1, \ldots, m$. Thus, by (7), for all $f \in C(\mathcal{M}, \mathbb{R})$,

$$\begin{aligned} \mathcal{L}^*(\nu)(f) &= \nu(\mathcal{L}(f)) = \sum_{j=1}^{m} \int_{D^j} \mathcal{L}(f)(x)d\nu(x) = \sum_{j=1}^{m} \int_{F^{-1}(D^j)} f(x)d\nu(x) \\ &= \int_{\mathcal{M}} f(x)d\nu(x) = \nu(f) \end{aligned}$$

that is, $\mathcal{L}^*(\nu) = \nu$.

Since $\varphi \in C^2(\mathcal{M}, \mathbb{R})$, by Remark 2.2.3 and by Lemma 2.1.3, the set

$$\Lambda_1 \stackrel{\text{def}}{=} \Lambda \cap \{f \in C(\mathcal{M}, \mathbb{R}) \ : \ \nu(f) = 1\}$$

is a non-empty closed convex set of $C(\mathcal{M}, \mathbb{R})$ and $\mathcal{L}(\Lambda_1) \subset \Lambda_1$. Now, we show that the operator $\widetilde{\mathcal{L}}$ has a fixed point in Λ_1. Then the existence and the properties of μ_F will follow by Theorem 2.1.6.

By Proposition 1.2.7, there is an integer $k_1 \geq 1$ such that $F^{k_1}(B(z, \epsilon_1)) = \mathcal{M}$ for all $z \in \mathcal{M}$. Let $L(k_1) \stackrel{\text{def}}{=} \sup_{y \in \mathcal{M}}\{\varphi_{k_1}(y)\}$ which is finite because $\varphi_{k_1} \in C^2(\mathcal{M})$. For $x, z \in \mathcal{M}$, there exists $y_1 \in F^{-k_1}(x) \cap B(z, \epsilon_1)$, and, if $f \in \Lambda_1$, then

$$\mathcal{L}^{k_1}(f)(x) = \sum_{y \in F^{-k_1}(x)} e^{\varphi_{k_1}(y)} f(y) \geq e^{\varphi_{k_1}(y_1)} f(y_1) \geq e^{-L(k_1) - C(z, y_1)} f(z).$$

Since $\mathcal{L}^*(\nu) = \nu$ and $\nu(f) = 1$, integration with respect to x yields

$$f(z) \leq e^{L(k_1) + C(z, y_1)} \nu(\mathcal{L}^{k_1}(f)) \leq e^{L(k_1) + L\epsilon_1} (\mathcal{L}^*)^{k_1}(\nu)(f) = e^{L(k_1) + L\epsilon_1},$$

that is, the set Λ_1 is bounded.

Take $f \in \Lambda_1$ and $x, x' \in \mathcal{M}$ such that $d(x, x') \leq \epsilon_1$. Then

$$|f(x) - f(x')| \leq e^{L(k_1) + L\epsilon_1} (e^{C(x, x')} - 1) \leq e^{L(k_1) + L\epsilon_1} (e^{Ld(x, x')} - 1)$$

and Λ_1 is equicontinuous.

Therefore, Λ_1 is a non-empty compact convex set of $C(\mathcal{M}, \mathbb{R})$ and $\mathcal{L}$ is a continuous map for which $\mathcal{L}(\Lambda_1) \subset \Lambda_1$. Hence, by the Schauder-Tychonoff theorem (see for example [Rud3] p.143), $\mathcal{L}$ has a fixed point $h \in \Lambda_1$.

The map h is strictly positive on $\mathcal{M}$. Otherwise, if $h(x) = 0$ for some $x \in \mathcal{M}$ then

$$\mathcal{L}^k(h)(x) = \sum_{y \in F^{-k}(x)} e^{\varphi_k(y)} h(y) = h(x) = 0.$$

Thus, since $e^{\varphi_k(y)} > 0$, $h(y) = 0$ for all $y \in \bigcup_{k \geq 0} F^{-k}(x)$ which, by (b) of Proposition 1.3.4, is dense in $\mathcal{M}$. Therefore, $h \equiv 0$ which does not belong to Λ_1 and we have a contradiction.

Now we prove that μ_F is equivalent to ν. Define the measure μ by setting $\mu(f) \stackrel{\text{def}}{=} \nu(fh)$ for all $f \in C(\mathcal{M}, \mathbb{R})$. Since $h > 0$, μ is equivalent to ν and $\frac{d\mu}{d\nu} = h$. Being $\mathcal{L}^*(\nu) = \nu$, then for $f \in C(\mathcal{M}, \mathbb{R})$

$$\mu(\widetilde{\mathcal{L}}(f)) = \nu(h\widetilde{\mathcal{L}}(f)) = \nu(\widetilde{\mathcal{L}}(fh)) = \mathcal{L}^*(\nu)(fh) = \nu(fh) = \mu(f).$$

Therefore, since $\mu(1) = \nu(h) = 1$,

$$\mu(f) = \lim_{k \to \infty} m(\widetilde{\mathcal{L}}^k(f)) = \mu(\mu_F(f)) = \mu_F(f),$$

that is, $\mu = \mu_F$.

(a) $\mu_F = \nu$ iff $h \equiv 1$.

(b) For $f \in C(\mathcal{M}, \mathbb{R})$, by (a) of Theorem 2.1.6,

$$\lim_{k \to \infty} \mathcal{L}^k(1) = \lim_{k \to \infty} h\widetilde{\mathcal{L}}^k(\frac{1}{h}) = h\mu_F(\frac{1}{h}) = h\nu(1) = h.$$

Thus, since $\mathcal{L}^*(\nu) = \nu$, we obtain

$$(\nu \circ F^{-k})(f) = \nu(f \circ F^k) = \nu(\mathcal{L}^k(f \circ F^k)) = \nu(f\mathcal{L}^k(1)) \to \nu(fh\nu(1)) = \mu_F(f),$$

that is, the sequence of measures $\{\nu \circ F^{-k}\}_{k \geq 0}$ converges weakly to μ_F. Q.E.D.

Remark 2.2.5 Rohlin's formula gives the entropy of $F \in \mathcal{E}(\mathcal{M})$ with respect to the measure μ_F (see [Ro]):

$$h_{\mu_F}(F) = \int_{\mathcal{M}} \log(|\det(D_x F)|) d\mu_F(x) > 0.$$

It is interesting to note that the two invariant measures that we have described, may coincide.

Theorem 2.2.6 [Wa1] *Let $F \in \mathcal{E}^1(\mathcal{M}) \cap C^2(\mathcal{M}, \mathcal{M})$ and assume that the normalized Lebesgue measure ν is F-invariant. Then ν is the unique measure of maximal entropy iff $|\det(D_x F)| = N$ for all $x \in \mathcal{M}$.*

As an application of the previous results we prove the following facts.

Corollary 2.2.7 *Let $F, G \in \mathcal{E}^1(\mathcal{M}) \cap C^2(\mathcal{M})$ and denote by μ_F and μ_G the normalized measures, respectively F-invariant and G-invariant, and equivalent to ν. If F and G commute then $\mu_F = \mu_G$.*

Proof. Let $\sigma \stackrel{\text{def}}{=} \mu_F \circ G^{-1} \in \text{Meas}_1(\mathcal{M})$. The measure σ is F-invariant because, if $E \in \text{Bor}(\mathcal{M})$, then

$$\sigma(F^{-1}(E)) = \mu_F(G^{-1}(F^{-1}(E))) = \mu_F(F^{-1}(G^{-1}(E))) = \mu_F(G^{-1}(E)) = \sigma(E).$$

Now we show that $\sigma \sim \nu$. If $E \in \text{Bor}(\mathcal{M})$ is such that $\nu(E) = 0$, since $\mu_G \sim \nu$, then

$$0 = \mu_G(E) = \mu_G(G^{-1}(E)) \text{ and } \nu(G^{-1}(E)) = 0.$$

But since $\mu_F \sim \nu$, then $0 = \mu_F(G^{-1}(E)) = \sigma(E)$. By Theorem 2.2.4, there is a unique F-invariant normalized measure equivalent to ν.

Thus $\mu_F = \sigma = \mu_F \circ G^{-1}$. This means that μ_F is G-invariant and as before we can conclude that $\mu_F = \mu_G$. Q.E.D.

Remark 2.2.8 Let $F \in \mathcal{E}^1(\mathcal{M}) \cap C^2(\mathcal{M})$. If V is non-empty open set of $\mathcal{M}$, by (c) of Proposition 1.3.4 and by (b) of Theorem 1.2.10, both

$$\mathrm{Do}_{\mathcal{M}}(F) \cap V \ \text{ and } \ \mathrm{Per}_{\mathcal{M}}(F) \cap V \ \text{ are dense in } V.$$

We will prove in a moment that $\nu(\mathrm{Do}_{\mathcal{M}}(F) \cap V) = \nu(V)$. On the other hand, M. Urbanski has proved in [Ur] that, if V is non-empty then the set $V \setminus \mathrm{Do}_{\mathcal{M}}(F)$, which contains the set $\mathrm{Per}_{\mathcal{M}}(F) \cap V$, has Hausdorff dimension equal to the dimension of the manifold $\mathcal{M}$.

Corollary 2.2.9 *Let $F \in \mathcal{E}^1(\mathcal{M}) \cap C^2(\mathcal{M})$. If $V \subset \mathcal{M}$ is an open set then $\nu(\mathrm{Do}_{\mathcal{M}}(F) \cap V) = \nu(V)$.*

Proof. It is enough to prove that $\nu(\mathrm{Do}_{\mathcal{M}}(F)) = 1$ If $\{V_i\}_{i \geq 1}$ is any countable base of open sets for the topology on $\mathcal{M}$, then

$$\mathrm{Do}_{\mathcal{M}}(F) = \bigcap_{i=1}^{\infty} (\bigcup_{k=0}^{\infty} F^{-k}(V_i)).$$

Letting $U_i \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} F^{-k}(V_i)$, then $F^{-j}(U_i) = \bigcup_{k=j}^{\infty} F^{-k}(V_i)) \subset U_i$ for all $j \geq 0$. Hence $F^{-j}(U_i) \cap U_i = F^{-j}(U_i)$ and, since μ_F is F-invariant,

$$\mu(U_i) = \mu_F(F^{-j}(U_i)) = \mu_F(F^{-j}(U_i)) \cap U_i \qquad \forall j \geq 0.$$

Since F is exact with respect to μ_F, by Remark 2.1.5, F is strong mixing and

$$\mu_F(U_i) = \lim_{j \to \infty} \mu_F(F^{-j}(U_i) \cap U_i) = (\mu_F(U_i))^2.$$

Since $\mu_F \sim \nu$, μ_F is positive on non-empty open sets, and thus $\mu_F(U_i) = \nu(U_i) = 1$. Now, it is enough to note that $\mathrm{Do}_{\mathcal{M}}(F) = \bigcap_{i=1}^{\infty} U_i$ and necessarily $\nu(\mathrm{Do}_{\mathcal{M}}(F)) = 1$. Q.E.D.

3

Set of periods

In this Chapter, we consider, for an expanding map F on a closed manifold $\mathcal{M}$, the *set of periods* $\mathcal{P}(F)$, i. e. the set of all positive integers m such that $p_F(m) > 0$ where

$$p_F(m) \stackrel{\text{def}}{=} \text{card}\{x \in \mathcal{M} \, : \, F^m(x) = x \text{ and } F^k(x) \neq x \text{ for } k = 1, \ldots, m-1\}.$$

First, by using the Conjugation Theorem, we show that the cardinality of the set of fixed points of F is equal to the *Nielsen number* $\mathcal{N}(F)$. In other words, all the fixed points are *essential* and no one of them can be eliminated by homotopy.

Afterwards, we prove that $\mathcal{P}(F)$ is cofinite, i. e. the set $\mathbb{N}^*/\mathcal{P}(F)$ is finite. This will be attained by establishing the following limit

$$\lim_{k\to\infty} \frac{\mathcal{N}(F^{k+1})}{\mathcal{N}(F^k)} = N,$$

where $N \geq 2$ is the degree of F, which will be computed exploiting the properties of the measure μ_0 of maximal entropy. Also a sufficient condition for being $\mathcal{P}(F) = \mathbb{N}^*$ is given.

Then, we investigate the *flat manifolds* which are those infra-nil-manifolds with the simply connected nilpotent group equal to $\mathbb{R}^n$. We give a suitable representation of a flat manifold $\mathcal{M}$ as the quotient space $\mathbb{R}^n/\Gamma$ where Γ is an *affine Bieberbach group*. After proving that flat manifolds admit always an expanding map, by the Conjugation Theorem, to each $F \in \mathcal{E}(\mathcal{M})$ we associate an endomorphism $\Phi_{(A,a)}$, where (A, a) is an affine map of $\mathbb{R}^n$ with

all eigenvalues of absolute value greater than 1. By the Nielsen number formula
$$\mathcal{N}(F) = \frac{1}{|\Psi|} \sum_{(U,u)\in\Psi} |\det(A-U)|,$$
where Ψ is the holonomy group of $\mathcal{M}$, we prove that on a flat manifold, the sets of periods of its expanding maps are uniformly cofinite. Moreover, the set of periods of a map F expanding on an infra-torus up to dimension 3 is always equal to $\mathbb{N}^*$. This results complete the investigation of the structure of $\mathcal{P}(F)$, at least for expanding maps, which was started by J. Llibre in the case of $\mathbf{T}^2$ and $\mathbf{T}^3$.

In the brief section, which ends the Chapter, we show how the set of the periods of an expanding maps on a nil-manifolds N/Z depends on the linearization of the nilpotent group N.

§3.1 Index theory and the Nielsen number.

Let X be a connected compact *polyhedron*. If U is an open set of X and ∂U is its boundary in X, we denote by $\mathcal{C}_U(X)$ the set of all continuous maps $F : X \to X$ without fixed points on ∂U.

It is possible to associate to each $F \in \mathcal{C}_U(X)$, an integer number $i(X, F, U)$, called the *index* of the map F on the open U. This index provides an algebraic computation of the cardinality of the set of fixed points of F contained in U and can be viewed as a generalization of the notion of multiplicity of zeros of a complex analytic functions of one variable.

In the modern development of this subject, the index is first defined abstractly as a function satisfying the following five axioms, which were inspired by the properties of the classical index:

(i) Localization: if $F, G \in \mathcal{C}_U(X)$ and $F(x) = G(x)$ for all $x \in U$ then
$$i(X, F, U) = i(X, G, U).$$

(ii) Homotopy invariance: If $H : X \times [0,1] \to X$ is a homotopy and

$F_t(\cdot) \stackrel{\text{def}}{=} H(\cdot, t)$ belongs to $\mathcal{C}_U(X)$ for all $t \in [0,1]$ then

$$i(X, F_0, U) = i(X, F_t, U) \qquad \forall t \in [0,1].$$

(iii) Additivity: let $F \in \mathcal{C}_U(X)$ and let $U_1, \ldots, U_s$ be a family of mutually disjoint open subsets of U such that the set $U \setminus \bigcup_{j=1}^{s} U_j$ does not contain any fixed point of F. Then $F \in \mathcal{C}_{U_j}(X)$ for $j = 1, \ldots, s$, and

$$i(X, F, U) = \sum_{j=1}^{s} i(X, F, U_j).$$

(iv) Normalization: if we denote by $L(X, F)$ the Lefschetz number of a continuous map $F : X \to X$ in X (see [Br1]), then

$$i(X, F, X) = L(X, F).$$

(v) Commutativity: if $F, G : X \to X$ are continuous maps such that $G \circ F \in \mathcal{C}_U(X)$, then $F \circ G \in \mathcal{C}_{G^{-1}(U)}(X)$ and

$$i(X, G \circ F, U) = i(X, F \circ G, G^{-1}(U)).$$

R. F. Brown has shown in [Br1] that such an index does exist and it is unique. Moreover, this index has an explicit expression when the map is differentiable. In fact, let $U \subset X \subset \mathbb{R}^n$ be an open set and let $F \in \mathcal{C}_U(X)$ be a C^1-map. If x_0 is the only fixed point of F in U and $\det(\mathbb{I} - D_{x_0}F) \neq 0$, then

$$i(\mathcal{M}, F, x_0) = \operatorname{sgn}(\det(\mathbb{I} - D_{x_0}F)). \tag{1}$$

Remark 3.1.1 Note that these axioms imply immediatly the following facts:

(a) if $i(X, F, U) \neq 0$, then F has a fixed point in U.

In fact, by additivity, $i(X, F, \emptyset) = 0$, and, if U contains no fixed point, then $i(X, F, U) = i(X, F, \emptyset) = 0$.

So, when x_0 is an isolated fixed point of F, we will write $i(X, F, x_0)$ instead of $i(X, F, U)$, where U is an open neighborhood of x_0 without any other fixed point in $\overline{U}$.

(b) If $L(X,F) \neq 0$ and G is homotopic to F then G has a fixed point in X. In fact, by normalization and homotopy invariance,

$$i(X,G,X) = L(X,G) = L(X,F) \neq 0$$

and (a) implies that G has a fixed point in X.

Lemma 3.1.2 [Ji] *Let $F : X \to X$ be a continuous map and define the following equivalence relation on* $\mathrm{Fix}_X(F)$*: two fixed points x_0 and x_1 are equivalent, written $x_0 \sim x_1$, iff there is a path $s : [0,1] \to \mathcal{M}$ from x_0 to x_1, such that $F \circ s$ and s are homotopic for a homotopy fixing endpoints.*

Then there is only a finite number of equivalence classes with respect to $\sim$.

Proof. We denote by $\mathbb{F}$ a generic equivalence class with respect to $\sim$. Now we prove that $\mathbb{F}$ is an open subset of the compact set $\mathrm{Fix}_X(F)$.

For $x_0 \in \mathrm{Fix}_X(F)$, we look for a neighborhood U of x_0, such that any fixed point $x_1 \in U$ belongs to the same class of x_0. Since X has a universal covering, there is a neighborhood W of x_0 such that every loop in W at x_0 is trivial in X and there is also a path-connected neighborhood U of x_0 such that $U \subset W \cap F^{-1}(W)$. If $x_1 \in U \cap \mathrm{Fix}_X(F)$, take a path s in U from x_0 to x_1. Then both s and $F \circ s$ are in W, and therefore are homotopic implying $x_0 \sim x_1$.

Therefore, since $\mathrm{Fix}_X(F)$ is compact, there is only a finite number of equivalence classes. Q.E.D.

Definition 3.1.3 By the preceding lemma, for any equivalence class $\mathbb{F}$ there exists an open set U such that $\mathbb{F} = U \cap \mathrm{Fix}_X(F)$, then we assign the index $i(X,F,U)$ to $\mathbb{F}$. The *Nielsen number* of F is defined as the number of the *essential fixed point classes*, i. e. the equivalence classes with non-zero index (for a general reference see the book of B. Jiang [Ji]).

The fundamental result which made so important the Nielsen number in the fixed point theory is the following.

Theorem 3.1.4 [Ji] *Let $F : X \to X$ be a continuous map. Then*

(a) $\mathcal{N}(F) \leq \mathrm{card}(\mathrm{Fix}_X(F))$;

(b) $\mathcal{N}(F) = \mathcal{N}(G)$ for all continuous maps G homotopic to F.

Proof. (a) Each essential fixed point class $\mathbb{F}$ contains at least a fixed point because $i(X, F, \mathbb{F}) \neq 0$.

(b) Let G be a continuous map homotopic to F and $H : X \times [0, 1] \to X$ the corresponding homotopy map. Any fixed point class $\mathbb{F}$ of F corresponds, via H, to a fixed point class $\mathbb{F}'$ of G. Then, by the homotopy invariance of the index, $i(X, F, \mathbb{F}) = i(X, G, \mathbb{F}')$. Therefore, the fixed point class $\mathbb{F}$ is essential iff also $\mathbb{F}'$ is essential. Q.E.D.

Remark 3.1.5 Note that, by the previous theorem,

$$\mathcal{N}(F) \leq \mathrm{MF}(F) \stackrel{\mathrm{def}}{=} \min\{\mathrm{card}(\mathrm{Fix}_X(G)) \quad : \quad G \text{ is homotopic to } F\}.$$

The polyhedron X is called *Wecken* if the equality $\mathcal{N}(F) = \mathrm{MF}(F)$ holds for all continuous map $F : X \to X$. It has been proved, see [Br2] as a general reference, that if X is any compact manifold of dimension $n \geq 3$ then it is a Wecken manifold. On the other hand, the surface consisting of a disc with two holes, often called the *pants surface*, is not Wecken.

§3.2 Nielsen numbers for expanding maps.

Now, we will see what happens when we choose F expanding on a closed manifold $\mathcal{M}$. By Corollary 1.3.8, the cardinality of the fixed point set is homotopy invariant for expanding maps, and therefore, by Theorem 3.1.4,

$$\mathcal{N}(F) = \mathcal{N}(G) \leq \mathrm{card}(\mathrm{Fix}_{\mathcal{M}}(G)) = \mathrm{card}(\mathrm{Fix}_{\mathcal{M}}(F))$$

for all $G \in \mathcal{E}(\mathcal{M})$ homotopic to F. The following theorem says that actually $\mathcal{N}(F) = \mathrm{card}(\mathrm{Fix}_{\mathcal{M}}(F))$.

Theorem 3.2.1 *If* $F \in \mathcal{E}(\mathcal{M})$ *then*

(a) $|i(\mathcal{M}, F, x)| = 1$ *for all* $x \in \mathrm{Fix}_{\mathcal{M}}(F)$*;*

(b) $|L(\mathcal{M}, F)| \leq \mathcal{N}(F) = \mathrm{card}(\mathrm{Fix}_{\mathcal{M}}(F))$.

Proof. (a) By Theorem 1.4.13 there exist an infra-nil-manifold $\mathcal{M}'$ and an endomorphism $\Phi \in \mathcal{E}^1(\mathcal{M}')$ such that F and Φ are topologically conjugate by some homeomorphism α_0: $\alpha_0 \circ F = \Phi \circ \alpha_0$. Therefore, if $x \in \mathcal{M}$ is a fixed point of F, then $\alpha_0(x)$ is the corresponding fixed point of Φ and, by the commutativity property of the index,

$$i(\mathcal{M}, F, x) = i(\mathcal{M}, \alpha_0^{-1} \circ (\alpha_0 \circ F), x) = i(\mathcal{M}', (\alpha_0 \circ F) \circ \alpha_0^{-1}, \alpha_0(x)) = i(\mathcal{M}', \Phi, \alpha_0(x)).$$

Since $\Phi \in \mathcal{E}^1(\mathcal{M}')$, the differential $D_x\Phi$ has all its eigenvalues with absolute value greater than 1. Then, by (1),

$$i(\mathcal{M}, \Phi, x) = i(\mathcal{M}', \Phi, \alpha_0(x)) = \mathrm{sgn}(\det(\mathbb{I} - D_{\alpha_0(x)}\Phi)) \neq 0.$$

(b) First we show that each equivalence class contains only one fixed point. Let x_0, x_1 be two equivalent fixed points and let $s : [0, 1] \to \mathcal{M}$ be a path from x_0 to x_1 such that $F \circ s \simeq s$. Let $\widetilde{F}$ be a lifting of F and $y_0 \in \pi^{-1}(x_0)$, then, by Remark 1.3.2, there is $\gamma_0 \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ such that $\widetilde{F}(y_0) = \gamma_0(y_0)$, that is $(\gamma_0^{-1} \circ \widetilde{F})(y_0) = y_0$.

So, $\widetilde{F}_0 = \gamma_0^{-1} \circ \widetilde{F}$ is another lifting of F, and since F is expanding, by Theorem 1.3.3, y_0 is the only fixed point of $\widetilde{F}$ on $\widetilde{\mathcal{M}}$. Let $\tilde{s}$ be the lifting of the path s such that $\tilde{s}(0) = y_0$. Then $\widetilde{F}_0 \circ \tilde{s}$ is a lifting of the path $F \circ s$ such that $(\widetilde{F}_0 \circ \tilde{s})(0) = y_0$. Hence, since $F \circ s \simeq s$, by the monodromy theorem their liftings $\widetilde{F}_0 \circ \tilde{s}$ and $\tilde{s}$, both starting from y_0, must have the same endpoint

$$y_1 = \tilde{s}(1) = (\widetilde{F}_0 \circ \tilde{s})(1) = \widetilde{F}_0(y_1).$$

Therefore, since the fixed point y_0 of $\widetilde{F}_0$ is unique, $y_0 = y_1$ and $x_1 = \pi(y_1) = \pi(y_0) = x_0$. Thus $\mathcal{N}(F)$ is the number of fixed points of F because, by (a), the index $i(\mathcal{M}, F, x)$ is non-zero for each $x \in \mathrm{Fix}_{\mathcal{M}}(F)$.

Moreover, since equivalence classes corrispond to fixed points, the first inequality in (b) follows directly from the normalization property:

$$|L(\mathcal{M}, F)| = |\sum_{x \in \mathrm{Fix}_{\mathcal{M}}(F)} i(\mathcal{M}, F, x)| \leq \sum_{x \in \mathrm{Fix}_{\mathcal{M}}(F)} |i(\mathcal{M}, F, x)| = \mathcal{N}(F).$$

Q.E.D.

Theorem 3.2.2 *If $F \in \mathcal{E}(\mathcal{M})$, then*

$$\lim_{k \to \infty} \frac{\mathcal{N}(F^{k+1})}{\mathcal{N}(F^k)} = N,$$

where $N \geq 2$ is the degree of F.

Proof. Let $\varphi \equiv -\log(N)$ and let $\mathcal{L}$ be the transfer operator associated to F and φ, then

$$\mathcal{L}^k(f)(x) = \widetilde{\mathcal{L}}^k(f)(x) = \frac{1}{N^k} \sum_{y \in F^{-k}(x)} f(y) \qquad \forall f \in C(\mathcal{M}, \mathbb{R})$$

and by (a) of Theorem 2.1.6, the sequence $\{\mathcal{L}^k(f)\}_{k \geq 1}$ converges uniformly to the costant map $\mu_0(f)$, where μ_0 is the normalized F-invariant measure with maximal entropy.

For $k \geq 1$, denote by μ_k the measure on $\mathcal{M}$ such that for all $x \in \mathcal{M}$

$$\mu_k(\{x\}) = \begin{cases} \frac{1}{N^k} & \text{if } F^k(x) = x \\ 0 & \text{otherwise} \end{cases} .$$

Now, we will prove that the sequence of measures $\{\mu_k\}_{k \geq 1}$ converges weakly to μ_0:

$$\lim_{k \to \infty} \mu_k(f) = \mu_0(f) \qquad \forall f \in C(\mathcal{M}, \mathbb{R}).$$

Actually, we can assume that $f \in C(\mathcal{M}, \mathbb{R})$ has its support, written $\mathrm{supp}(f)$, such that $\mathrm{supp}(f) \subset\subset D$, where D is a compact set with $\mathrm{diam}(D) \leq \epsilon_1$. If $a \in D$ then it suffices to show that

$$\lim_{k \to \infty} (\mu_k(f) - \mathcal{L}^k(f)(a)) = 0.$$

By Theorem 1.2.7, if $k \geq 1$, $F^{-k}(D)$ is the disjoint union of the sets $D_i^{(k)}$ with $\mathrm{diam}(D_i^{(k)}) \leq \frac{\epsilon_1}{\lambda^k}$ for $i = 1, \ldots, N^k$, where $\lambda > 1$ is the expanding constant of F. Let k be so large, that

$$\frac{\epsilon_1}{\lambda^k} \leq \inf\{d(x, x') : x \in \mathrm{supp}(f),\ x' \notin D\}.$$

Let $\mathcal{I}_k \stackrel{\text{def}}{=} \{1 \leq i \leq N^k : D_i^{(k)} \cap \mathrm{supp}(f) \neq \emptyset\}$. Then, as a consequence of our choice of k, if $i \in \mathcal{I}_k$, then $D_i^{(k)} \subset D$. Moreover, by Theorem 1.3.3, the homeomorphism $F_i^{-k} : D \to D_i^{(k)}$ is a $\frac{1}{\lambda^k}$-contraction. Hence F_i^{-k} has a unique fixed point, say $z_i^{(k)}$, in $D_i^{(k)}$ for $i \in \mathcal{I}_k$.

Letting $a_i^{(k)} \stackrel{\text{def}}{=} F^{-k}(a) \cap D_i^{(k)}$, we have

$$\lim_{k\to\infty} |\mu_k(f) - \mathcal{L}^k(f)(a)| \leq \lim_{k\to\infty} \frac{1}{N^k} \sum_{i\in\mathcal{I}_k} |f(z_i^{(k)}) - f(a_i^{(k)})| = 0$$

because $d(z_i^{(k)}, a_i^{(k)}) \leq \frac{\epsilon_1}{\lambda^k}$ and f is uniformly continuous on $\mathcal{M}$.

Then, when f is the costant map 1,

$$\lim_{k\to\infty} \mu_k(1) = \mu_0(\mathcal{M}) = 1,$$

and, since $\frac{\mathcal{N}(F^k)}{N^k} = \mu_k(1)$ for all $k \geq 1$,

$$\lim_{k\to\infty} \frac{\mathcal{N}(F^{k+1})}{\mathcal{N}(F^k)} = \lim_{k\to\infty} \frac{\mathcal{N}(F^{k+1})}{N^{k+1}} \frac{N^k}{\mathcal{N}(F^k)} N = N.$$

Q.E.D.

§3.3 Cofiniteness of the set of periods.

Definition 3.3.1 For $m \geq 1$, the number of periodic points of period m for a map $F : \mathcal{M} \to \mathcal{M}$ is denoted by

$$p_F(m) \stackrel{\text{def}}{=} \mathrm{card}\{x \in \mathcal{M} \ : \ F^m(x) = x \text{ and } F^k(x) \neq x \text{ for } k = 1, \ldots, m-1\}.$$

The *set of periods* $\mathcal{P}(F)$ of the map F is the set of positive integers m such that $p_F(m) > 0$.

Remark 3.3.2 We already know, by Theorem 1.2.10, that if $F \in \mathcal{E}(\mathcal{M})$, then, for all $m \geq 1$, $p_F(m)$ is finite and the set $\mathrm{Per}_{\mathcal{M}}(F)$ is infinite. Therefore, $\mathcal{P}(F)$ is infinite.

Moreover, since $\mathcal{N}(F^m) = \sum_{k|m} p_F(k)$ for all $m \geq 1$, i. e. $\mathcal{N}(F^m)$ is the *Möbius transform* of $p_F(m)$, then

$$p_F(m) = \sum_{k|m} \vartheta(\frac{m}{k})\mathcal{N}(F^k) = \sum_{k|m} \vartheta(k)\mathcal{N}(F^{\frac{m}{k}}) \qquad \forall m \geq 1, \tag{2}$$

where, for $k \geq 1$,

$$\vartheta(k) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } k = 1 \\ (-1)^r & \text{if } k \text{ is product of } r \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

(see [Hu] p. 108). Note that for $m > 1$, $\sum_{k|m} \vartheta(k) = 0$.

The following theorem tells us that, actually, $\mathcal{P}(F)$ is *cofinite*, i. e. $\mathbb{N}^* \setminus \mathcal{P}(F)$ is a finite set.

Theorem 3.3.3 *If $F \in \mathcal{E}(\mathcal{M})$, then $\mathcal{P}(F)$ is cofinite. More precisely, there is an integer $k_0 \geq 1$ such that*

$$\frac{\mathcal{N}(F^{k+1})}{\mathcal{N}(F^k)} \geq \frac{1+\sqrt{5}}{2} \qquad \forall k \geq k_0 \tag{3}$$

and $m \in \mathcal{P}(F)$ for all $m \geq 2k_0$. If $k_0 = 1$ then $\mathcal{P}(F) = \mathbb{N}^*$.

Proof. By Theorem 3.2.2,

$$\lim_{k\to\infty} \frac{\mathcal{N}(F^{k+1})}{\mathcal{N}(F^k)} = N \geq 2 > \frac{1+\sqrt{5}}{2}.$$

Thus there exists an integer $k_0 \geq 1$ such that (3) holds.

Now we prove that (3) implies that $m \in \mathcal{P}(F)$ for all $m \geq 2k_0$. Note that, for $m \geq 4$,

$$\begin{aligned} \sum_{2\leq k\leq \frac{m}{2}} \mathcal{N}(F^{m-k}) + \sum_{k|m,k<m} \vartheta(\frac{m}{k})\mathcal{N}(F^k) &= \sum_{k|m,2\leq k\leq\frac{m}{2}} (\mathcal{N}(F^{m-k}) + \vartheta(\frac{m}{k})\mathcal{N}(F^k)) \\ &\quad + \sum_{k\nmid m,2\leq k\leq\frac{m}{2}} \mathcal{N}(F^{m-k}) + \vartheta(m)\mathcal{N}(F). \end{aligned}$$

For $2 \le k \le \frac{m}{2}$, $\mathcal{N}(F^{m-k}) \ge \mathcal{N}(F^k) \ge 1$ when $k|m$, and $\mathcal{N}(F^{m-k}) \ge \mathcal{N}(F) \ge 1$. Therefore, since $|\vartheta(\cdot)| \le 1$,

$$\sum_{2\le k\le \frac{m}{2}} \mathcal{N}(F^{m-k}) + \sum_{k|m, k<m} \vartheta(\frac{m}{k})\mathcal{N}(F^k) \ge \sum_{k|m, 2\le k\le \frac{m}{2}} (1+\vartheta(\frac{m}{k})) + [(\sum_{k\not| m, 2\le k\le \frac{m}{2}} 1) + \vartheta(m)] \ge 0. \quad (4)$$

Let $\tau = \frac{1+\sqrt{5}}{2}$. By (2), (3) and (4) for $m \ge \max(4, 2k_0)$

$$\begin{aligned} p_F(m) &= \mathcal{N}(F^m) + \sum_{k|m,k<m} \vartheta(\frac{m}{k})\mathcal{N}(F^k) \ge \mathcal{N}(F^m) - \sum_{2\le k\le \frac{m}{2}} \mathcal{N}(F^{m-k}) \\ &\ge \mathcal{N}(F^m) - \sum_{\frac{m}{2}\le k\le m-2} \mathcal{N}(F^k) \ge \mathcal{N}(F^m)(1 - \sum_{\frac{m}{2}\le k\le m-2} \frac{1}{\tau^{m-k}}) \\ &> 1 - \sum_{k=2}^{\infty} \frac{1}{\tau^k} = \frac{\tau^2-\tau-1}{\tau(\tau-1)} \ge 0. \end{aligned}$$

whence $m \in \mathcal{P}(F)$.

To complete the proof, it suffices to check the case when $k_0 = 1$ and $1 \le m \le 3$. Since $\mathcal{N}(F) \ge 1$, then $1 \in \mathcal{P}(F)$. Moreover,

$$\begin{aligned} p_F(2) &= \mathcal{N}(F^2) - \mathcal{N}(F) \ge (\tau - 1)\mathcal{N}(F) > 0, \\ p_F(3) &= \mathcal{N}(F^3) - \mathcal{N}(F) \ge (\tau^2 - 1)\mathcal{N}(F) > 0 \end{aligned}$$

show that also $2, 3 \in \mathcal{P}(F)$. Q.E.D.

§3.4 Flat manifolds.

By Theorem 1.4.13, we know that, if $\mathcal{E}(\mathcal{M}) \ne \emptyset$, then $\mathcal{M}$ is homeomorphic to a infra-nil-manifold. However, given an infra-nil-manifold $\mathcal{M} = \mathsf{N}/\Gamma$, $\mathcal{E}(\mathcal{M})$ can be empty. M. Shub and D. Epstein in [EpSh] proved that, when the nilpotent group N is the abelian group $\mathbb{R}^n$, then $\mathcal{E}(\mathcal{M})$ is always non-empty.

Definition 3.4.1 A cocompact, torsion free, discrete subgroup Γ of $O(n)\ltimes\mathbb{R}^n$, the group of the affine isometries of $\mathbb{R}^n$, is called a *Bieberbach group* and $\mathcal{M} = \mathbb{R}^n/\Gamma$ is the *flat manifold* associated to Γ.

Remark 3.4.2 The term *flat* derives from the fact that the flat manifolds are connected compact riemannian manifolds whose Levi-Civita connection has identically vanishing curvature (see [Ch] and [Wo] as general references).

Moreover, by Theorem 1.4.7 (see [Au]):

(a) $\Gamma \cap (\{\mathbb{I}\} \ltimes \mathbb{R}^n)$ is a finitely generated free abelian group isomorphic to $\mathbb{Z}^n$ and is the unique normal abelian subgroup of Γ.

(b) $\Psi = \Gamma/(\Gamma \cap (\{\mathbb{I}\} \ltimes \mathbb{R}^n)) \simeq r(\Gamma)$ is the *holonomy group* of the manifold and is finite.

The following proposition will allow us to find a better representation of a given flat manifold.

Proposition 3.4.3 *Let Γ be Bieberbach group. Then, there is an element (B, b) of the affine group* $\mathrm{Aff}(\mathbb{R}^n)$*, which conjugates Γ to a subgroup $\Gamma' \subset$* $\mathrm{Aff}(\mathbb{R}^n)$*, such that for any $\gamma \in \Gamma'$:*

$$\gamma = (U, u) \quad \textit{with} \quad U \in GL(n, \mathbb{Z}) \quad \textit{and} \quad |\Psi| u \in \mathbb{Z}^n.$$

The group Γ' is called affine Bieberbach group.

Proof. By (a), there exists $B \in \mathrm{GL}(n, \mathbb{R})$ such that the rotational parts of the elements of the group $\Gamma'' \stackrel{\text{def}}{=} (B, 0)\Gamma(B, 0)^{-1}$ have integer coefficients.

Moreover, by (b), the group $\Psi'' \stackrel{\text{def}}{=} \Gamma''/(\{\mathbb{I}\} \ltimes \mathbb{Z}^n)$ is finite with the same cardinality $|\Psi|$ of Ψ and

$$\Psi'' = \{(U_i, u_i) \ : \ i = 1, \ldots, |\Psi|\} \subset \mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n.$$

Now, for all positive integers $i, j \leq |\Psi|$ there exists a positive integer $k \leq |\Psi|$ such that $U_i U_j = U_k$ and

$$t((U_i, u_i)(U_j, u_j)) = U_i u_j + u_i \equiv u_k \pmod{\mathbb{Z}^n}.$$

Fix i and sum the congruences varying j over $\{1, \ldots, |\Psi|\}$. Since, also k varies over $\{1, \ldots, |\Psi|\}$, we obtain that there is $p_i \in \mathbb{Z}^n$ such that

$$U_i(\sum_{j=1}^{|\Psi|} u_j) + |\Psi| u_i = (\sum_{j=1}^{|\Psi|} u_j) + p_i.$$

Letting $b \stackrel{\text{def}}{=} -\frac{1}{|\Psi|}\sum_{j=1}^{|\Psi|} u_j$, then $-U_i b + u_i = -b + \frac{1}{|\Psi|}p_i$ and

$$(\mathbb{I}, b)(U_i, u_i)(\mathbb{I}, b)^{-1} = (U_i, b + u_i - U_i b) = (U_i, \frac{1}{|\Psi|}p_i).$$

Therefore, the wanted affine trasformation is (B, b). Q.E.D.

Remark 3.4.4 Let $\mathcal{M} = \mathbb{R}^n/\Gamma$ be a flat manifold and let $\mathcal{M}'$ be the quotient space $\mathbb{R}^n/\Gamma'$, where $\Gamma' = (B, b)\Gamma(B, b)^{-1}$ is the affine Bieberbach group of the previous proposition. Then, a map $f : \mathcal{M} \to \mathcal{M}'$ is well defined such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{R}^n & \stackrel{(B,b)}{\longrightarrow} & \mathbb{R}^n \\ \pi \downarrow & & \downarrow \pi' \\ \mathcal{M} & \stackrel{f}{\longrightarrow} & \mathcal{M}' \end{array}$$

and, moreover it is easy to check that f is a homeomorphism. This fact allows us to consider in our future investigations, a flat manifold as the quotient space of $\mathbb{R}^n$ by an affine Bieberbach group rather than a Bieberbach group.

Moreover, by Proposition 3.4.3, an affine Bieberbach group Γ is totally determined by a *reduced system of representatives*

$$\mathcal{R}(\Gamma) \stackrel{\text{def}}{=} \{(U_i, u_i) \ : \ i = 1, \ldots, m\}$$

where $U_i \in \mathrm{GL}(n, \mathbb{Z})$ and $u_i = \frac{1}{|\Psi|}p_i$ with $p_i \in \{0, \ldots, |\Psi| - 1\}$ for $i = 1, \ldots, m \leq |\Psi|$. The group Γ is generated by the elements of the set $\mathcal{R}(\Gamma) \cup (\{\mathbb{I}\} \ltimes \mathbb{Z}^n)$.

As there are only finitely many finite subgroups of $\mathrm{GL}(n, \mathbb{Z})$, then, there is only a finite number of reduced systems for affine Bieberbach groups in dimension n. Hence, there is also a finite number of n-dimensional flat manifolds.

Now, we collect some rudiments of cohomology theory of groups that we will need later.

Let G be an arbitrary group acting on $\mathbb{R}^n$. The transform of an element $u \in \mathbb{R}^n$ by $g \in G$ is denoted by $g \cdot u \in \mathbb{R}^n$. For $j \geq 1$, let $\mathcal{C}^j(G, \mathbb{R}^n)$ be the set of functions $c : G^j \to \mathbb{R}^n$, and let $\mathcal{C}^0(G, \mathbb{R}^n) \stackrel{\text{def}}{=} \mathbb{R}^n$. Each $\mathcal{C}^j(G, \mathbb{R}^n)$ is a group and it is called the set of *j-cochains* for G in $\mathbb{R}^n$. For $j \geq 0$, we denote by $\delta^j : \mathcal{C}^j(G, \mathbb{R}^n) \to \mathcal{C}^{j+1}(G, \mathbb{R}^n)$ the *j-coboundary operator*:

$$\begin{aligned}(\delta^j c)(g_1, \cdots, g_{j+1}) \quad &= \quad g_1 \cdot c(g_2, \cdots, g_{j+1}) + \sum_{i=1}^{j} (-1)^i c(g_1, \cdots, g_i g_{i+1}, \cdots, g_{j+1}) \\ &\quad\; + (-1)^{j+1} c(g_1, \cdots, g_j).\end{aligned}$$

Let $Z^j(G, \mathbb{R}^n) \stackrel{\text{def}}{=} \mathrm{Ker}(\delta^j)$ and $B^j(G, \mathbb{R}^n) \stackrel{\text{def}}{=} \mathrm{Im}(\delta^{j-1})$ be respectively the group of *j-cocycles* and *j-coboundaries*. For $j \geq 1$, since $\delta^j \circ \delta^{j-1} \equiv 0$, $B^j(G, \mathbb{R}^n)$ is a subgroup of the abelian group $Z^j(G, \mathbb{R}^n)$ and we can define $H^j(G, \mathbb{R}^n) = Z^j(G, \mathbb{R}^n)/B^j(G, \mathbb{R}^n)$, that is the *$j$-cohomology group* of G with coefficients in $\mathbb{R}^n$.

Let H be a normal subgroup of G which acts trivially on $\mathbb{R}^n$. Hochschild and Serre investigated the relations between the cohomology groups of G, H and G/H and in particular they proved the following result.

Theorem 3.4.5 [HoSe] *The following exact sequence holds:*

$$0 \to H^1(G/H, \mathbb{R}^n) \to H^1(G, \mathbb{R}^n) \stackrel{res}{\to} H^1(H, \mathbb{R}^n)^G \to H^2(G/H, \mathbb{R}^n) \to H^2(G, \mathbb{R}^n). \tag{5}$$

The homomorphism res *is defined as the restriction of a cocycle $c \in \mathcal{C}^1(G, \mathbb{R}^n)$ to H and $H^1(H, \mathbb{R}^n)^G$ is the set of all elements c in $H^1(H, \mathbb{R}^n)$ such that $g \cdot c(g^{-1}hg) = c(h)$ for all $g \in G$ and $h \in H$.*

In our case $G \stackrel{\text{def}}{=} \Gamma$ and $H \stackrel{\text{def}}{=} \{\mathbb{I}\} \ltimes \mathbb{Z}^n$; so $G/H = \Psi$. The action of Γ on $\mathbb{R}^n$ is defined by:

$$\gamma \cdot u = t(\gamma(\mathbb{I}, u)\gamma^{-1}) = r(\gamma)u \qquad \forall \gamma \in \Gamma \;\text{ and }\; \forall u \in \mathbb{R}^n.$$

The finiteness of the holonomy group Ψ implies the next lemma.

Lemma 3.4.6 *The cohomology group* $H^1(\Psi, \mathbb{R}^n) = 0$.

Proof. If $c \in Z^1(\Psi, \mathbb{R}^n)$, i. e. $(\delta^1 c) = 0$, then

$$c(\gamma_1\gamma_2) = \gamma_1 \cdot c(\gamma_2) + c(\gamma_1) \qquad \forall \gamma_1, \gamma_2 \in \Psi.$$

If $d \stackrel{\text{def}}{=} \frac{1}{|\Psi|} \sum_{\gamma\in\Psi} c(\gamma) \in \mathbb{R}^n$, summing over $\gamma_2 \in \Gamma$ and dividing by $|\Psi|$, the order of the group Ψ, we obtain $d = \gamma_1 \cdot d + c(\gamma_1)$, that is $c \in B^1(\Psi, \mathbb{R}^n)$. Thus, $H^1(\Psi, \mathbb{R}^n) = 0$. Q.E.D.

A fundamental property of affine Bieberbach groups is expressed by the following theorem.

Theorem 3.4.7 [LSY] *Let* Γ *be an affine Bieberbach group. If* $\varphi : \Gamma \to \Gamma$ *is an injective homomorphism, there exists* $(A, a) \in \mathbb{Z}^{n\times n} \ltimes \mathbb{R}^n \subset \mathrm{Aff}(\mathbb{R}^n)$ *such that, for all* $\gamma = (U, u) \in \Gamma$*:*

$$\varphi(\gamma) = (A, a)^\sharp(\gamma) = (A, a)\gamma(A, a)^{-1} = (AUA^{-1}, Au + (I - AUA^{-1})a). \quad (6)$$

Proof. By (a) of Remark 3.4.2, $\{\mathbb{I}\} \ltimes \mathbb{Z}^n$ is the unique maximal normal abelian subgroup of Γ. Thus it is invariant by φ, and the injective homomorphism induced by φ on $\mathbb{Z}^n$ extends uniquely to an automorphism of $\mathbb{R}^n$ represented by a matrix $A \in \mathbb{Z}^{n\times n}$. Hence

$$\varphi((\mathbb{I}, p)) = (\mathbb{I}, Ap) \qquad \forall p \in \mathbb{Z}^n. \quad (7)$$

For any $\gamma = (U, u) \in \Gamma$ and $p \in \mathbb{Z}^n$, applying φ to both sides of

$$\gamma(\mathbb{I}, p)\gamma^{-1} = (\mathbb{I}, Up) \text{ to get } \varphi(\gamma)(\mathbb{I}, Ap)(\varphi(\gamma))^{-1} = (\mathbb{I}, AUp),$$

yields $r(\varphi(\gamma)) = AUA^{-1}$.

Now, define the map $c : \Gamma \to \mathbb{R}^n$

$$c(\gamma) \stackrel{\text{def}}{=} A^{-1}t(\varphi(\gamma)) - t(\gamma) \qquad \forall \gamma \in \Gamma. \quad (8)$$

We say that $c \in Z^1(\Gamma, \mathbb{R}^n)$. In fact, by the definition of c, for $\gamma_1 = (U_1, u_1), \gamma_2 = (U_2, u_2) \in \Gamma$

$$\begin{cases} t(\varphi(\gamma_1\gamma_2)) = Ac(\gamma_1\gamma_2) + AU_1u_2 + Au_1 \\ t(\varphi(\gamma_1)\varphi(\gamma_2)) = AU_1A^{-1}(Ac(\gamma_2) + Au_2) + Ac(\gamma_1) + Au_1 \end{cases}$$

and these two equations imply that $c(\gamma_1\gamma_2) = \gamma_1 \cdot c(\gamma_2) + c(\gamma_1)$.

Moreover, the exact sequence (5) and Lemma 3.4.6 implies the following exact sequence:

$$0 = H^1(\Psi, \mathbb{R}^n) \to H^1(\Gamma, \mathbb{R}^n) \stackrel{\text{res}}{\to} H^1(\{\mathbb{I}\} \ltimes \mathbb{R}^n, \mathbb{R}^n)^{\Gamma}.$$

Hence the homomorphism res is injective whereas, by (7) and (8),

$$\text{res}(c)((\mathbb{I}, p)) = c((\mathbb{I}, p)) = A^{-1}t((\mathbb{I}, Ap)) - t((\mathbb{I}, p)) = 0 \qquad \forall p \in \mathbb{Z}^n.$$

Therefore $c \in B^1(\Gamma, \mathbb{R}^n)$ and there exists $d \in \mathbb{R}^n$ such that $c(\gamma) = \gamma \cdot d - d$ for all $\gamma \in \Gamma$. If $a \stackrel{\text{def}}{=} -Ad$, then, by (8), for any $\gamma = (U, u) \in \Gamma$

$$t(\varphi(\gamma)) = Ac(\gamma) + At(\gamma) = Au + (\mathbb{I} - AUA^{-1})a.$$

Thus

$$\varphi(\gamma) = (r(\varphi(\gamma)), t(\varphi(\gamma))) = (AUA^{-1}, Au + (\mathbb{I} - AUA^{-1})a).$$

Q.E.D.

Remark 3.4.8 Let $\mathcal{M} = \mathbb{R}^n/\Gamma$ be a flat manifold. Since $\mathbb{I} \ltimes \mathbb{Z}^n$ is a subgroup of the affine Bieberbach group Γ, then $\mathcal{M}$ is always covered by the *torus* $\mathbf{T}^n \stackrel{\text{def}}{=} \mathbb{R}^n/(\mathbb{I} \ltimes \mathbb{Z}^n)$. When this covering is not trivial, i. e. when $\Psi \neq \{1\}$, the manifold is called an *infra-torus*.

Moreover, given an injective homomorphism $\varphi : \Gamma \to \Gamma$, there is map (A, a), that is a lifting on $\mathbb{R}^n$ of a map $\Phi_{(A,a)} : \mathcal{M} \to \mathcal{M}$, which induces on Γ a homomorphism $\Phi^{\sharp}_{(A,a)}$ equal to φ. Then, the following commutative diagram holds

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{(A,a)} & \mathbb{R}^n \\ \pi' \downarrow & & \downarrow \pi' \\ \mathbf{T}^n & \xrightarrow{R_a \circ \Phi_A} & \mathbf{T}^n \\ \pi'' \downarrow & & \downarrow \pi'' \\ \mathcal{M} & \xrightarrow{\Phi_{(A,a)}} & \mathcal{M} \end{array}$$

where: $\pi = \pi'' \circ \pi'$, $R_a : \mathbf{T}^n \to \mathbf{T}^n$ is a *toral rotation*,

$$R_a(x) = ((\pi'(a))_1 x_1, \ldots, (\pi'(a))_n x_n) \qquad \forall x \in \mathbf{T}^n$$

and $\Phi_A : \mathbf{T}^n \to \mathbf{T}^n$ is a *toral linear endomorphism*

$$\Phi_A(x) = (x_1^{a_{1,1}} \cdots\ x_n^{a_{1,n}}, \ldots, x_1^{a_{n,1}} \cdots\ x_n^{a_{n,n}}) \qquad \forall x \in \mathbf{T}^n.$$

The map $\Phi_{(A,a)}$ is an endomorphism of $\mathcal{M}$ and since the differential of (A,a) at 0 is A, then, by Remark 1.4.11, $\Phi_{(A,a)}$ is differentiably expanding iff $\mathrm{sp}(A) \cap \overline{\Delta} = \emptyset$, where $\mathrm{sp}(A)$ is the *spectrum* of the matrix A and $\overline{\Delta}$ the closed unit disc in $\mathbb{C}$.

Now we are ready to establish the result of Epstein and Shub. We give a new proof which is much simpler than the original one.

Theorem 3.4.9 [EpSh] *If $\mathcal{M}$ is a flat manifold, then $\mathcal{E}^1(\mathcal{M})$ is not empty.*

Proof. The flat manifold $\mathcal{M}$ can be represented as quotient space $\mathbb{R}^n/\Gamma$ where Γ is an affine Bieberbach group. The affine map $((|\Psi|+1)\mathbb{I}, 0)$ induces on Γ a homomorphism φ such that, for all $\gamma = (U,u) \in \Gamma$,

$$\varphi(\gamma) = ((|\Psi|+1)\mathbb{I}, 0)\gamma((|\Psi|+1)\mathbb{I}, 0)^{-1} = (U, (|\Psi|+1)u) = (U,u)(\mathbb{I}, |\Psi|u).$$

By Proposition 3.4.3, $|\Psi|u \in \mathbb{Z}^n$, hence $\varphi(\gamma) \in \Gamma$. Therefore, the affine map $((|\Psi|+1)\mathbb{I}, 0)$ induces also an endomorphism $\Phi_{(|\Psi|+1)I,0)}$ which belongs to $\mathcal{E}^1(\mathcal{M})$ because $(|\Psi|+1) \geq 2$. Q.E.D.

The next lemma will be useful later.

Lemma 3.4.10 *If $A \in \mathbb{Z}^{n\times n}$ is a non-singular matrix then:*

(a) Φ_A is a self-covering of $\mathbf{T}^n$ with degree equal to $|\det(A)| \geq 1$;

(b) if sp(A) has no roots of unity, i. e. $\det(A^k - \mathbb{I}) \neq 0$ for all $k \geq 1$ then

$$\mathrm{card}(\mathrm{Fix}_{\mathbf{T}^n}(\Phi_A^k)) = |\det(A^k - \mathbb{I})| \qquad \forall k \geq 1.$$

Proof. (a) Since A is non-singular, Φ_A is a local homeomorphism of $\mathbf{T}^n$ and, by Theorem 1.1.1, is a self-covering of $\mathbf{T}^n$.

Moreover there exist two matrices $P, Q \in \mathrm{GL}(n, \mathbb{Z})$ such that $A = PDQ$ where $D \in \mathbb{Z}^{n\times n}$ is diagonal (see [Hu] p. 384). This means that $\Phi_A = \Phi_P \circ \Phi_D \circ \Phi_Q$ where Φ_P and Φ_Q are homeomorphisms. Hence

$$\mathrm{card}(\Phi_A^{-1}(1)) = \mathrm{card}(\Phi_D^{-1}(1)) = |\det(D)| = |\det(A)| \geq 1,$$

where 1 is the identity element of the group $\mathbf{T}^n$.

(b) A point $x \in \mathrm{Fix}_{\mathbf{T}^n}(\Phi_A^k)$ iff there exists $y \in \mathbb{R}^n$ such that $(A^k - \mathbb{I})y \in \mathbb{Z}^n$. Therefore, since $\det(A^k - \mathbb{I}) \neq 0$ for all $k \geq 1$, by (a),

$$\mathrm{Fix}_{\mathbf{T}^n}(\Phi_A^k) = \mathrm{Ker}(\Phi_{(A^k - I)}) = \Phi_{(A^k-I)}^{-1}(1) = |\det(A^k - \mathbb{I})|.$$

Q.E.D.

The following theorem gives a formula for the computation of the Nielsen number for an expanding map of a flat manifold (see [Mc] as general reference for the computation of Nielsen numbers).

Theorem 3.4.11 *Let $\mathcal{M}$ be a flat manifold. If $F \in \mathcal{E}(\mathcal{M})$ and $\Phi_{(A,a)}$ is the endomorphism associated to F, then*

$$\mathcal{N}(F) = \frac{1}{|\Psi|} \sum_{(U,u)\in\Psi} |\det(A - U)|. \tag{9}$$

All the values of $\det(A - U)$ for $(U, u) \in \Psi$ are different from zero and have the same sign.

Proof. Since $\mathcal{N}(F) = \mathcal{N}(\Phi_{(A,a)})$, it is enough to compute the number of fixed points of $\Phi_{(A,a)}$. Since $x \in \mathrm{Fix}_{\mathcal{M}}(\Phi_{(A,a)})$ iff there exist $y \in \mathbb{R}^n$ and $(U,u) \in \Gamma$ such that $(A,a)(y) = (U,u)(y)$ and $\pi(y) = x$, then, by Remark 3.4.8,

$$\mathrm{Fix}_{\mathcal{M}}(\Phi_{(A,a)}) = \pi\Big(\bigcup_{(U,u)\in\Gamma} (A-U)^{-1}(u-a)\Big) = \pi''\Big(\bigcup_{(U,u)\in\Psi} \Phi^{-1}_{(A-U)}(\pi'(u-a))\Big) \tag{10}$$

Now, we show that if (U,u) and (V,v) are two different elements of Ψ then

$$\Phi^{-1}_{(A-U)}(\pi'(u-a)) \cap \Phi^{-1}_{(A-V)}(\pi'(v-a)) = \emptyset.$$

Otherwise there exists $y \in \mathbb{R}^n$ and $p, q \in \mathbb{Z}^n$ such that

$$\begin{cases} (A-U)y = u - a + p \\ (A-V)y = v - a + q \end{cases}$$

thus $(V, v+q)^{-1}(U, u+p)y = y$. Since, by Remark 1.3.2, the action of Γ on $\mathbb{R}^n$ is properly discontinuous, then $(V, v+q)^{-1}(U, u+p) = (\mathbb{I}, 0)$ and $U = V$ contradicting the hypothesis. By (10), since the degree of the covering π'' is equal to $|\Psi|$, then, by Theorem 3.2.1,

$$\mathcal{N}(\Phi_{(A,a)}) = \frac{1}{|\Psi|} \sum_{(U,u)\in\Psi} \mathrm{card}(\Phi^{-1}_{(A-U)}(\pi'(u-a))).$$

To complete the proof, in view of the preceding lemma, it is enough to prove that all the numbers $\det(A-U)$, for $(U,u) \in \Gamma$, are different from zero and have the same sign. So, $\mathrm{card}(\Phi^{-1}_{(A-U)}(\pi'(u-a))) = |\det(A-U)|$.

By definition, the affine Bieberbach group Γ is conjugate to a subgroup of affine isometries of $\mathbb{R}^n$. Hence there exists a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that $\|U\| \leq 1$ for all $(U,u) \in \Gamma$.

If we assume that $\det(A-\mathbb{I})\det(A-U) \leq 0$ then, by the continuity of the real function $t \to \det(A-(1-t)\mathbb{I}-tU)$, there is a $t_0 \in [0,1]$ such that $\det(A-(1-t_0)\mathbb{I}-t_0U) = 0$. That is $Ay = (1-t_0)y + t_0Uy$ for a non-zero $y \in \mathbb{R}^n$. Since $\Phi_{(A,a)} \in \mathcal{E}^1(\mathcal{M})$, $\mathrm{sp}(A) \cap \overline{\Delta} = \emptyset$, and there is some integer $k \geq 2$ such that $\|A^k y\| > 1$ and $\|A^{k-1}y\| \leq 1$. Moreover, since $(A,a)^\sharp(\Gamma) \subset \Gamma$, by (6), $A^{k-1}U = VA^{k-1}$ for some $(V,v) \in \Gamma$.

Thus, gathering all these informations, we have

$$1 < \|A^k y\| = \|(1-t_0)A^{k-1}y + t_0 A^{k-1}Uy\| \leq \|(1-t_0)\mathbb{I} + t_0 V\| \|A^{k-1}y\| \leq 1,$$

hence a contradiction. Q.E.D.

§3.5 Set of periods on flat manifolds.

Given a flat manifold $\mathcal{M}$ and a map $F \in \mathcal{E}(\mathcal{M})$, by Theorem 3.3.3, we already know that $\mathcal{P}(F)$ is cofinite. However, some periods may be missing in general. For example, $\Phi_{-2I} \in \mathcal{E}(\mathbf{T}^n)$ has no points of period 2, because, by (2),

$$p_{\Phi_{-2I}}(2) = \mathcal{N}(\Phi_{-2I}^2) - \mathcal{N}(\Phi_{-2I}) = |\det(4\mathbb{I} - \mathbb{I})| - |\det(-2\mathbb{I} - \mathbb{I})| = 3^n - 3^n = 0.$$

Actually, the finite set of missing periods of an expanding map on $\mathbf{T}^n$ can be quite large.

Proposition 3.5.1 [ABLSS] *There exists an expanding map F on $\mathbf{T}^n$ such that*

$$\mathcal{P}(F) \subset \mathbb{N}^* \setminus \{k > 1 : k | (n+1)\}.$$

Proof. Take the toral endomorphism Φ_E where E is the $n \times n$ integer matrix having $Q(z) = z^n + 2z^{n-1} + 2z^{n-2} + \cdots + 2z + 2$ as its characteristic polynomial.

Since $Q(z)(z-1) = z^n(z+1) - 2$, if $a \in \overline{\Delta}$ is a zero of Q then $|a|^n|a+1| = 2$ and therefore $a = 1$. But $Q(1) \neq 0$, hence $\mathrm{sp}(E) \cap \overline{\Delta} = \emptyset$ and $\Phi_E \in \mathcal{E}^1(\mathbf{T}^n)$.

Clearly $\mathcal{N}(\Phi_E^{n+1}) \geq \mathcal{N}(\Phi_E^k) \geq \mathcal{N}(\Phi_E)$, when $k|(n+1)$. Since $\det(E) = 2$, the equality $E^n(E + \mathbb{I}) = 2\mathbb{I}$ implies that $\det(E + \mathbb{I}) = 1$ and

$$\mathcal{N}(\Phi_E^{n+1}) = |\det(E^{n+1} - \mathbb{I})| = |\det(E^{n+1}(E+\mathbb{I}) - (E+\mathbb{I})| = |\det(2E - E - \mathbb{I})| = \mathcal{N}(\Phi_E).$$

Thus, if $k|(n+1)$ then $\mathcal{N}(\Phi_E^k) = \mathcal{N}(\Phi_E)$. By Remark 3.3.2, for $m \geq 2$, we have

$$p_{\Phi_E}(m) = (\sum_{k|m} \vartheta(\frac{m}{k}))\mathcal{N}(\Phi_E) = 0.$$

Q.E.D.

The question arises whether for a given flat manifold $\mathcal{M}$ the sets of periods of the expanding maps are *uniformly cofinite*:

$$\bigcap\{\mathcal{P}(F) \ : \ F \in \mathcal{E}(\mathcal{M})\} \quad \text{is cofinite.}$$

As regards $\mathbf{T}^n$, uniform cofiniteness has been already proven recently by B. Jiang and J. Llibre in [JiLl] by using the following lemma on algebraic numbers due to M. Mignotte.

Lemma 3.5.2 [JiLl] *Let α be a nonzero algebraic number with minimal polynomial $Q \in \mathbb{Z}[x]$ of degree d. If $|\alpha| \neq 1$ then*

$$|1 - |\alpha|| \geq \frac{1}{2^{d^2} M(\alpha)^d}$$

with $M(\alpha) \stackrel{\text{def}}{=} |a| \prod_{i=1}^{d} \max\{1, |\alpha_i|\}$ where a is the leading coefficient of Q and $\alpha_1, \ldots, \alpha_d$ the roots of Q.

Now, we generalize, for the expanding maps, the result of B. Jiang and J. Llibre to a generic flat manifold.

Theorem 3.5.3 *Let $\mathcal{M}$ be a flat manifold then the sets of periods of the expanding maps are uniformly cofinite.*

Proof. Let n be the dimension of $\mathcal{M}$, choose $F \in \mathcal{E}(\mathcal{M})$ and let $\Phi_{(A,a)}$ be the endomorphism associated to F. Suppose that $\alpha_1, \ldots, \alpha_n$ are the eigenvalues of A and let $\varrho(A) \stackrel{\text{def}}{=} \max\{|\alpha_1|, \ldots, |\alpha_n|\}$, i. e. $\varrho(A)$ is the *spectral radius* of A. If $V \in r(\Psi)$ and $k \geq 1$ then

$$(A^k V)^m = (A^k V A^{-k})(A^{2k} V A^{-2k}) \ldots (A^{mk} V A^{-mk}) A^{mk} \quad \forall m \geq 1.$$

Since $Ar(\Psi)A^{-1} \subset r(\Psi)$ and $r(\Psi)$ is a finite group of order $|\Psi|$, there is an integer $1 \leq m_0 \leq |\Psi|$ such that $A^{m_0 k} V A^{-m_0 k} = V$. Therefore $(A^k V)^{m_0} =$

UA^{km_0} for some $U \in r(\Psi)$ and $(A^kV)^{m_0|\Psi|} = U^{|\Psi|}A^{km_0|\Psi|} = A^{km_0|\Psi|}$. This means that, in absolute value, the eigenvalues of A^kV and A^k are the same: $|\alpha_1|^k, \dots, |\alpha_n|^k$. Since $|\det(U)| = 1$ for all $U \in r(\Psi)$, then by (9),

$$\mathcal{N}(F^k) = \frac{1}{|\Psi|} \sum_{(U,u)\in\Psi} |\det(A^k - U)| = \frac{1}{|\Psi|} \sum_{(V,v)\in\Psi} |\det(A^kV - \mathbb{I})|.$$

Therefore, for $m, k \geq 1$

$$\frac{\mathcal{N}(F^m)}{\mathcal{N}(F^k)} = \frac{\sum_{(V,v)\in\Psi} |\det(A^mV - \mathbb{I})|}{\sum_{(V,v)\in\Psi} |\det(A^kV - \mathbb{I})|} \geq \prod_{i=1}^{n} \frac{|\alpha_i|^m - 1}{|\alpha_i|^k + 1}. \tag{11}$$

The eigenvalues $\alpha_1, \dots, \alpha_n$ are algebraic numbers greater than 1 in absolute value: the minimal polynomial of each α_i has degree $d_i \leq n$ and $1 < M(\alpha_i) \leq \varrho(A)^n$. Hence, by the previous lemma,

$$|\alpha_i| - 1 \geq \frac{1}{2^{d_i^2} M(\alpha_i)^{d_i}} \geq \frac{1}{(2\varrho(A))^{n^2}}. \tag{12}$$

Let $1 \leq k < m$ be such that k divides m. If $|\alpha_i| \geq 2$ then

$$\frac{|\alpha_i|^m - 1}{|\alpha_i|^k + 1} \geq |\alpha_i|^{m-k} \frac{|\alpha_i|^k - 1}{|\alpha_i|^k + 1} \geq \frac{1}{3}.$$

If $1 < |\alpha_i| < 2$ then, by (12),

$$\frac{|\alpha_i|^m - 1}{|\alpha_i|^k + 1} \geq \frac{|\alpha_i| - 1}{3} \geq \frac{1}{3(2\varrho(A))^{n^2}}.$$

Therefore, since $k \leq \frac{m}{2}$, by (11),

$$\frac{\mathcal{N}(F^m)}{\mathcal{N}(F^k)} \geq \left(\frac{1}{3(2\varrho(A))^{n^2}}\right)^{n-1} \frac{\varrho(A)^m - 1}{\varrho(A)^k + 1} \geq \frac{\varrho(A)^{\frac{m}{2}} - 1}{(6\varrho(A))^{n^3}}. \tag{13}$$

Assume that $m \geq 2n^3$. Then there is $R > 1$ such that the right member of the above inequality is an increasing function with respect to $\varrho(A)$ for $\varrho(A) > R$. Hence, by (13), there is $m_0 \geq 2n^3$ such that the inequality

$$\frac{\mathcal{N}(F^m)}{\mathcal{N}(F^k)} \geq \frac{R^{\frac{m}{2}} - 1}{(6R)^{n^3}} > \frac{m}{2}$$

holds for all $m \geq m_0$. So, by (2),

$$p_F(m) \geq \mathcal{N}(F^m) - \sum_{k|m, k<m} \mathcal{N}(F^k) > \mathcal{N}(F^m)(1 - \sum_{k|m, k\leq \frac{m}{2}} \frac{2}{m}) \geq 0,$$

that is $m \in \mathcal{P}(F)$ for $m \geq m_0$.

In the remaining case, when $1 < \varrho(A) \leq R$, there is only a finite number of possibilities for the spectrum of A because the coefficients of the characteristic polynomial of A are integers bounded by $n!R^n$. Let r be the minimum of the lengths of the eigenvalues of such spectra, then $r > 1$. By (11), there is $k_0 \geq 1$ such that for $k \geq k_0$

$$\frac{\mathcal{N}(F^{k+1})}{\mathcal{N}(F^k)} \geq |\det(A)| \prod_{i=1}^{n} \frac{1 - (\frac{1}{|\alpha_i|})^{k+1}}{1 + (\frac{1}{|\alpha_i|})^k} \geq 2 \left(\frac{1 - (\frac{1}{r})^{k+1}}{1 + (\frac{1}{r})^k} \right)^n \geq \frac{1 + \sqrt{5}}{2}.$$

Therefore, by Theorem 3.3.3, $m \in \mathcal{P}(F)$ for $m \geq 2k_0$. Q.E.D.

Remark 3.5.4 For $F \in \mathcal{E}(\mathbf{T}^1)$, $\mathcal{P}(F) \subset \mathbb{N}^* \setminus \{2\}$. In fact, first of all, $p_{\Phi_{-2}}(2) = 0$. Now, let $m \geq 3$. Since m does not divide $m - 1$, it is easily checked that, if $|A| \geq 2$ then

$$p_{\Phi_A}(m) \geq |A^m - 1| - \sum_{k=1}^{m-2} |A^k - 1| \geq |A| - 1 > 0.$$

For $F \in \mathcal{E}(\mathbf{T}^2)$ the possible missing periods are 2, 3 and 4 (see [ABLSS] and [Ha]), whereas for $F \in \mathcal{E}(\mathbf{T}^3)$, they are 2, 3, 4, 5 and 6 (see [JiLl]). This situation is summarized in the following table, which has been carried out during an experimental investigation using *Mathematica*™.

Torus	Characteristic Polynomial of A	$\mathbb{N}^*/\mathcal{P}(\Phi_A)$
$\mathbf{T}^1$	$x + 2$	2
$\mathbf{T}^2$	$x^2 + 2x + 2$	$2, 3$
	$x^2 + 2$	4
$\mathbf{T}^3$	$x^3 + 2$	$2, 6$
	$x^3 - 2$	3
	$x^3 + x^2 + x + 2$	$2, 4$
	$x^3 + x^2 + 2$	5

On the other hand, we will show using the Nielsen number formula (9), that for F expanding on an n-infra-torus for $n \leq 3$, $\mathcal{P}(F) = \mathbb{N}^*$ (for the Klein bottle see [Ll]).

Now we give a complete list of the n-infra-tori for $n \leq 3$ (see for example [BBNWZ] or [Wo]). As we have noted in Remark 3.4.4, we can characterize each flat manifold $\mathcal{M} = \mathbb{R}^n/\Gamma$ by the reduced system of representatives $\mathcal{R}(\Gamma)$ of the corresponding affine Bieberbach group Γ. The elements of $\mathcal{R}(\Gamma)$, together with the integer traslations $\{\mathbb{I}\} \ltimes \mathbb{Z}^n$, generate the group Γ.

Of course, there are no infratori in dimension 1, whereas, the *Klein bottle* $\mathbf{K}^2 = \mathbb{R}^2/\mathcal{B}_1$ is the only 2-infratorus. This is the reduced system of the affine Bieberbach group $\mathcal{B}_1$ and the holonomy group:

$$\mathcal{R}(\mathcal{B}_1) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \right) \right\}, \ \Psi \simeq \mathbb{Z}_2.$$

If $n = 3$ there are 9 infra-tori $\mathcal{M} = \mathbb{R}^3/\Gamma$. We list the respective reduced systems and the holonomy groups There are 5 orientable infra-tori with the affine Bieberbach groups $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$, $\mathcal{B}_5$ and $\mathcal{B}_6$:

$$(1)\ \mathcal{R}(\mathcal{B}_2) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \right) \right\}, \ \Psi \simeq \mathbb{Z}_2;$$

$$(2)\ \mathcal{R}(\mathcal{B}_3) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \end{bmatrix} \right) \right\}, \ \Psi \simeq \mathbb{Z}_3;$$

$$(3)\ \mathcal{R}(\mathcal{B}_4) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} \\ 0 \\ 0 \end{bmatrix} \right) \right\}, \ \Psi \simeq \mathbb{Z}_4;$$

$$(4)\ \mathcal{R}(\mathcal{B}_5) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{6} \\ 0 \\ 0 \end{bmatrix} \right) \right\}, \ \Psi \simeq \mathbb{Z}_6;$$

$$(5)\ \mathcal{R}(\mathcal{B}_6) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right) \right\}, \Psi \simeq \mathbb{Z}_2 \times$$

$\mathbb{Z}_2$.

Moreover, there are 4 non-orientable infra-tori with the associated affine Bieberbach groups $\mathcal{B}_7$, $\mathcal{B}_8$, $\mathcal{B}_9$, $\mathcal{B}_{10}$:

$(1')$ $\mathcal{R}(\mathcal{B}_7) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \right) \right\}$, $\Psi \simeq \mathbb{Z}_2$;

$(2')$ $\mathcal{R}(\mathcal{B}_8) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \right) \right\}$, $\Psi \simeq \mathbb{Z}_2$;

$(3')$ $\mathcal{R}(\mathcal{B}_9) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \right), \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \right) \right\}$, $\Psi \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$;

$(4')$ $\mathcal{R}(\mathcal{B}_{10}) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \right), \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} \right) \right\}$, $\Psi \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

The following theorem classifies completly the set of expanding maps on the 10 infra-tori up to dimension 3.

Proposition 3.5.5 *If $\mathcal{M} = \mathbb{R}^n/\Gamma$ is an infra-torus such that $n \leq 3$ and $F \in \mathcal{E}(\mathcal{M})$, then F is topologically conjugate to a endomorphism $\Phi_{(A,a)}$ such that* $\mathrm{sp}(A) \cap \overline{\Delta} = \emptyset$ *and (A,a) belongs to the corresponding conjugacy class $\Theta(\Gamma) \subset \mathbb{Z}^{n\times n} \ltimes \mathbb{R}^n$ hereunder defined:*

$$\Theta(\mathcal{B}_1) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 2s_1+1 & 0 \\ 0 & s_2 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{r_1}{2} \end{bmatrix} \right) \right\};$$

$$\Theta(\mathcal{B}_2) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 2s_1+1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{r_1}{2} \\ \frac{r_2}{2} \end{bmatrix} \right) \right\};$$

$$\Theta(\mathcal{B}_3) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 3s_1+1 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a+b \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{r_1}{3} \\ \frac{r_2}{3} \end{bmatrix} \right), \left(\begin{bmatrix} 3s_1+2 & 0 & 0 \\ 0 & a & b \\ 0 & a+b & -a \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{r_1}{3} \\ \frac{r_2}{3} \end{bmatrix} \right) : \frac{r_1+r_2}{3} \in \mathbb{Z} \right\};$$

$$\Theta(\mathcal{B}_4) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 4s_1+1 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{r_1}{2} \\ \frac{r_2}{2} \end{bmatrix} \right), \left(\begin{bmatrix} 4s_1+3 & 0 & 0 \\ 0 & a & b \\ 0 & b & -a \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{r_1}{2} \\ \frac{r_2}{2} \end{bmatrix} \right) : \tfrac{r_1+r_2}{2} \in \mathbb{Z} \right\};$$

$$\Theta(\mathcal{B}_5) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 6s_1+1 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a-b \end{bmatrix}, \begin{bmatrix} 0 \\ r_1 \\ r_2 \end{bmatrix} \right), \left(\begin{bmatrix} 6s_1+5 & 0 & 0 \\ 0 & a & b \\ 0 & b-a & -a \end{bmatrix}, \begin{bmatrix} 0 \\ r_1 \\ r_2 \end{bmatrix} \right) \right\};$$

$$\Theta(\mathcal{B}_6) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} \frac{r_1}{2} \\ \frac{r_2}{2} \\ \frac{r_3}{2} \end{bmatrix} \right), \left(\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{bmatrix}, \begin{bmatrix} \frac{r_1}{2}+\frac{1}{4} \\ \frac{r_2}{2} \\ \frac{r_3}{2} \end{bmatrix} \right), \left(\begin{bmatrix} 0 & b & 0 \\ a & 0 & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} \frac{r_1}{2} \\ \frac{r_2}{2} \\ \frac{r_3}{2}+\frac{1}{4} \end{bmatrix} \right), \left(\begin{bmatrix} 0 & 0 & c \\ a & 0 & 0 \\ 0 & b & 0 \end{bmatrix}, \begin{bmatrix} \frac{r_1}{2} \\ \frac{r_2}{2}+\frac{1}{4} \\ \frac{r_3}{2}+\frac{1}{4} \end{bmatrix} \right), \left(\begin{bmatrix} 0 & b & 0 \\ 0 & 0 & c \\ a & 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{r_1}{2}+\frac{1}{4} \\ \frac{r_2}{2}+\frac{1}{4} \\ \frac{r_3}{2} \end{bmatrix} \right), \left(\begin{bmatrix} 0 & 0 & c \\ 0 & b & 0 \\ a & 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{r_1}{2}+\frac{1}{4} \\ \frac{r_2}{2}+\frac{1}{4} \\ \frac{r_3}{2}+\frac{1}{4} \end{bmatrix} \right) : a, b, c \text{ are odd} \right\};$$

$$\Theta(\mathcal{B}_7) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{r_1}{2} \end{bmatrix} \right) : a \text{ is odd, } c \text{ is even} \right\};$$

$$\Theta(\mathcal{B}_8) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} 2s_1+1 & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{bmatrix}, \begin{bmatrix} 0 \\ r_1 \\ 0 \end{bmatrix} \right) \right\};$$

$$\Theta(\mathcal{B}_9) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{r_1}{2} \\ \frac{r_2}{2} \end{bmatrix} \right) : a, c \text{ are odd} \right\};$$

$$\Theta(\mathcal{B}_{10}) \stackrel{\text{def}}{=} \left\{ \left(\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{r_1}{2} \\ \frac{r_2}{2} \end{bmatrix} \right) : a, b, c \text{ are odd} \right\}$$

where the coefficients $r_1, r_2, r_3 \in \mathbb{Z}$.

Proof. Since F induces an injective homomorphisms $\widetilde{F}^\sharp$, by Theorem 3.4.7, it suffices to classify all the maps $(A, a) \in \mathrm{Aff}(\mathbb{R}^n)$, for $n = 2, 3$, such that

$A \in \mathbb{Z}^{n\times n}$, $\mathrm{sp}(A) \cap \overline{\Delta} = \emptyset$

$$(AUA^{-1}, Au + (I - AUA^{-1})a) \in \Gamma \qquad \forall (U,u) \in \Gamma$$

and induce different homomorphisms on Γ. For this purpose, it is helpful to start by finding all the automorphisms of the finite group Ψ.

We give the complete proof only for the Klein bottle. As regards the other cases, the proof can be developed in a similar way.

When $\mathcal{M} = \mathbf{K}^2$, then $\Psi \simeq \mathbb{Z}_2$ and therefore the only automorphism of Ψ is the identity. Hence

$$A \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and this means that A is diagonal. Then (6) holds iff there exists $s_1, r_1 \in \mathbb{Z}$ such that

$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + (\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} s_1 + \frac{1}{2} \\ r_1 \end{bmatrix}.$$

Solving the above linear system we find that $a_{11} = 2s_1 + 1$ and $a_2 = \frac{r_1}{2}$. We can let $a_1 = 0$ because it has no restrictions. So $\Theta(\mathcal{B}_1)$ has been determined. Q.E.D.

Using the above classification, we can establish the following result on the set of periods.

Theorem 3.5.6 *Let $\mathcal{M} = \mathbb{R}^n/\Gamma$ be an infra-torus with $n \leq 3$. If $F \in \mathcal{E}(\mathcal{M})$ and if $\Phi_{(A,a)}$ is the associated endomorphism with $(A,a) \in \Theta(\Gamma)$, then the Nielsen number $\mathcal{N}(F^k)$ of the k-iterate of F for $k \geq 1$, in terms of the entries a_{ij} and the minors A_{ij} of A, is given by:*

(a) if $\Gamma = \mathcal{B}_1$, $\mathcal{N}(F^k) = |a_{11}^k - 1||a_{22}|^k$;

(b) if $\Gamma \in \{\mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5\}$, $\mathcal{N}(F^k) = |a_{11}^k - 1||\det(A_{11})^k + 1|$;

(c) if $\Gamma = \mathcal{B}_6$, $\mathcal{N}(F^k) = |\det(A)^k - 1|$;

(d) if $\Gamma = \mathcal{B}_7$, $\mathcal{N}(F^k) = |a_{33}|^k|\det(A_{33}^k - \mathbb{I})|$;

(e) if $\Gamma = \mathcal{B}_8$, $\mathcal{N}(F^k) = |a_{11}^k - 1||\det(A_{11})^k - 1|$.
(f) if $\Gamma \in \{\mathcal{B}_9, \mathcal{B}_{10}\}$, $\mathcal{N}(F^k) = |a_{11}^k - 1||\det(A_{11})|^k$.
Moreover, $\mathcal{P}(F) = \mathbb{N}^*$.

Proof. On each manifold $\mathcal{M} = \mathbb{R}^n/\Gamma$, the Nielsen number $\mathcal{N}(F^k) = \mathcal{N}(\Phi^k_{(A,a)})$, with $(A, a) \in \Theta(\Gamma)$ and $k \geq 1$, is computed by (9). Also here, we give the complete proof only for the Klein bottle. As regards the other cases, the proof can be developed in a similar way.

When $\mathcal{M} = \mathbf{K}^2$, A is a diagonal matrix and

$$\mathcal{N}(\Phi^k_{(A,a)}) = \frac{1}{2}|(a_{11}^k - 1)(a_{22}^k - 1) + (a_{11}^k - 1)(a_{22}^k + 1)| = |a_{11}^k - 1||a_{22}|^k.$$

Now, going back to the general case, we prove that in every case $\mathcal{P}(F) = \mathbb{N}^*$. By Theorem 3.3.3, it is enough to prove that $\frac{\mathcal{N}(F^{k+1})}{\mathcal{N}(F^k)} \geq \frac{1+\sqrt{5}}{2}$ for all $k \geq 1$.

(a) If $\Gamma = \mathcal{B}_1$ then

$$\frac{\mathcal{N}(F^{k+1})}{\mathcal{N}(F^k)} \geq \frac{|a_{11}|^{k+1} - 1}{|a_{11}|^k + 1}|a_{22}| \geq (|a_{11}| - 1)|a_{22}| \geq 4$$

because $|a_{11}| \geq 3$ and $|a_{22}| \geq 2$.

(b) and (e) If $\Gamma \in \{\mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_8\}$ then $|a_{11}| \geq 2$, $|\det(A_{11})| \geq 2$ and

$$\frac{\mathcal{N}(F^{k+1})}{\mathcal{N}(F^k)} \geq \frac{|a_{11}|^{k+1} - 1}{|a_{11}|^k + 1} \frac{|\det(A_{11})|^{k+1} - 1}{|\det(A_{11})|^k + 1} \geq (|a_{11}| - 1)(|\det(A_{11})| - 1) \geq 2.$$

To complete the proof in this case, it suffices to exclude the case when $|a_{11}| = |\det(A_{11})| = 2$. If $|a_{11}| = 2$ then $\Gamma = \mathcal{B}_3$ and $|\det(A_{11})| = |a^2+b^2+ab|$ Assume that also $|\det(A_{11})| = 2$: this implies that the integers a and b are even and positive. Therefore $|a| \geq 2$, $|b| \geq 2$ and the following inequality holds

$$2 = |a^2 + b^2 + ab| \geq a^2 + b^2 - |ab| = (|a| - |b|)^2 + |ab| \geq 4$$

which is a contradiction.

(c) If $\Gamma = \mathcal{B}_6$ then

$$\frac{\mathcal{N}(F^{k+1})}{\mathcal{N}(F^k)} \geq \frac{|\det(A)|^{k+1}-1}{|\det(A)|^k+1} \geq |\det(A)|-1 \geq 2$$

because $|\det(A)| \geq 3$.

(d) If $\Gamma = \mathcal{B}_7$ then $\mathcal{N}(F^k) = |a_{33}|^k \mathcal{N}(\Phi^k_{A_{33}})$ with $|a_{33}| \geq 2$.

Clearly $1 \in \mathcal{P}(F)$. Moreover, for $m \geq 2$,

$$p_F(m) = \sum_{k|m} \vartheta(\frac{m}{k})\mathcal{N}(F^k) \geq (|a_{33}|^m - \sum_{k=1}^{m-1} |a_{33}|^k)\mathcal{N}(\Phi^m_{A_{33}}) \geq 1$$

because $|\vartheta(\frac{m}{k})| \leq 1$ and $\mathcal{N}(\Phi^m_{A_{33}}) \geq \mathcal{N}(\Phi^k_{A_{33}}) \geq 1$ for $k|m$. Hence $m \in \mathcal{P}(F)$.

(f) If $\Gamma \in \{\mathcal{B}_9, \mathcal{B}_{10}\}$ then

$$\frac{\mathcal{N}(F^{k+1})}{\mathcal{N}(F^k)} \geq \frac{|a_{11}|^{k+1}-1}{|a_{11}|^k+1}|\det(A_{11})| \geq (|a_{11}|-1)|\det(A_{11})| \geq 12$$

because $|a_{11}| \geq 3$ and $|\det(A_{11})| \geq 6$. Q.E.D.

Remark 3.5.7 For arbitrary dimension $n > 3$, the set of periods of a map F expanding on a n-infra-torus is not always equal to $\mathbb{N}^\star$. Here it is an example. Let $\mathcal{M} = \mathbb{R}^6/\Gamma$ be the infra-torus with the following reduced system and holonomy group:

$\mathcal{R}(\Gamma) = \left\{\left(\begin{bmatrix} \mathbb{I}_2 & O \\ O & -\mathbb{I}_4 \end{bmatrix}, \frac{1}{2}e_1\right)\right\}, \ \Psi \simeq \mathbb{Z}_2.$

Choose the endomorphism $\Phi_{(A,a)} \in \mathcal{E}(\mathcal{M})$ where

$$(A,a) = \left(\begin{bmatrix} A_+ & O \\ O & A_- \end{bmatrix}, 0\right) \text{ with } A_+ = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} \text{ and } A_- = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \end{bmatrix}.$$

Now we prove that the endomorphism $\Phi_{(A,a)}$ has no points of period 3.

By (9), for all $k \geq 1$

$$\mathcal{N}(\Phi^k_{(A,a)}) = \frac{1}{2}|\det(A^k_+ - \mathbb{I}_2)|(|\det(A^k_- - \mathbb{I}_4)| + |\det(A^k_- + \mathbb{I}_4)|). \tag{14}$$

Since the characteristic polynomials of A_+ and A_- are respectively z^2+z+2 and z^4+z^2+2 then

$$A_+^2+A_++\mathbb{I}_2=-\mathbb{I}_2 \quad \text{and} \quad (A_-^2+A_-+\mathbb{I}_4)(A_-^2-A_-+\mathbb{I}_4)=-\mathbb{I}_4.$$

Hence $|\det(A_+^3-\mathbb{I}_2)| = |\det(A_+-\mathbb{I}_2)|$ because $|\det(A_+^2+A_++\mathbb{I}_2)| = 1$, moreover $|\det(A_-^2+A_-+\mathbb{I}_4)| = |\det(A_-^2-A_-+\mathbb{I}_4)| = 1$. Therefore, by (14),

$$\begin{aligned} p_{\Phi_{(A,a)}}(3) &= \mathcal{N}(\Phi_{(A,a)}^3)-\mathcal{N}(\Phi_{(A,a)}) \\ &= \frac{1}{2}|\det(A_+-\mathbb{I}_2)|\Big[(|\det(A_-^3-\mathbb{I}_4)|-|\det(A_--\mathbb{I}_4)|) \\ &\quad + (|\det(A_-^3+\mathbb{I}_4)|-|\det(A_-+\mathbb{I}_4)|)\Big] \\ &= \frac{1}{2}|\det(A_+-\mathbb{I}_4)|\Big[|\det(A_--\mathbb{I}_4)|(|\det(A_-^2+A_-+\mathbb{I}_4)|-1) \\ &\quad + |\det(A_-+\mathbb{I}_4)|(|\det(A_-^2-A_-+\mathbb{I}_4)|-1)\Big] = 0. \end{aligned}$$

§3.6 Set of periods on nil-manifolds.

Suppose now, that $\mathcal{M}=\mathsf{N}/\mathsf{Z}$ is a nil-manifold and define inductively the series $\mathsf{N}_{i+1} \overset{\text{def}}{=} [\mathsf{N}_i,\mathsf{N}]$ for $i\geq 0$, where $\mathsf{N}_0=\mathsf{N}$. Since N is nilpotent this series is finite with length, say $l\geq 1$. For $\mathsf{Z}_i \overset{\text{def}}{=} \mathsf{Z}\cap\mathsf{N}_i$ for $i=0,\dots,l$, each group $\mathsf{Z}_{i-1}/\mathsf{Z}_i$ is abelian and torsion-free . This set of groups represents the so-called *linearization* of Z and, as we can see from the next theorem, it makes it possible to the compute Nielsen number of a continuous map $F:\mathcal{M}\to\mathcal{M}$. In fact, D. V. Anosov has proved in [An] the following theorem that generalizes a previous result established for the torus (see [BBPT]).

Theorem 3.6.1 [An] *If $\mathcal{M}=\mathsf{N}/\mathsf{Z}$ is a nil-manifold and $F:\mathcal{M}\to\mathcal{M}$ is a continuous map, then*

$$\mathcal{N}(F)=|L(\mathcal{M},F)|=\prod_{i=1}^{l}|\det(A_i-\mathbb{I})|,$$

where each A_i is the $n_i\times n_i$ integer matrix induced by the homomorphism $\widetilde{F}^\sharp$ of Z on $\mathsf{Z}_{i-1}/\mathsf{Z}_i$.

Theorem 3.6.2 *If $\mathcal{M} = \mathsf{N}/\mathsf{Z}$ is a nil-manifold and $F \in \mathcal{E}(\mathcal{M})$, then*

$$\mathcal{P}(F) \supset \bigcup_{i=1}^{l} \mathcal{P}(\Phi_{A_i}).$$

Moreover, the sets of periods of the expanding maps are uniformly cofinite and

$$\bigcap\{\mathcal{P}(F) \ : \ F \in \mathcal{E}(\mathcal{M})\} \supset \bigcap\{\mathcal{P}(F') \ : \ F' \in \mathcal{E}(\mathbf{T}^{n'})\}$$

where $n' = \min\{n_1, \ldots, n_l\}$.

Proof. By the previous theorem, $\mathcal{P}(F) = \mathcal{P}(\Phi_A)$ where the endomorphism Φ_A belongs to $\mathcal{E}(\mathbf{T}^n)$, n is the dimension of $\mathcal{M}$ and A is the $n \times n$ integer matrix which has on the main diagonal the blocks A_i for $i = 1, \ldots, l$.

Now, if $m \in \bigcup_{i=1}^{l} \mathcal{P}(\Phi_{A_i})$, there exists $x_{i_0} \in \mathbf{T}^{n_{i_0}}$ for some i_0, which is a periodic point of period m for the endomorphism $\Phi_{A_{i_0}}$. Therefore, if x_i is a fixed point of Φ_{A_i} for $i \in \{1, \ldots, l\} \setminus \{i_0\}$ then $x = (x_1, \ldots, x_{i_0}, \ldots, x_l) \in \mathbf{T}^n$ is a periodic point of period m for the endomorphism Φ_A, i.e. $m \in \mathcal{P}(\Phi_A) = \mathcal{P}(F)$. Hence, the uniform cofiniteness of the sets of periods follows directly from Theorem 3.5.3. Q.E.D.

Corollary 3.6.3 *Let $\mathcal{M}$ be the nil-manifold $UT_r(\mathbb{R})/UT_r(\mathbb{Z})$ for $r \geq 2$ then*

$$\bigcap\{\mathcal{P}(F) \ : \ F \in \mathcal{E}(\mathcal{M})\} = \mathbb{N}^* \setminus \{2\}.$$

Proof. In this case, we have that $l = r - 1$ and for $i = 1, \ldots, r - 1$,

$$\mathsf{Z}_i = [\mathrm{UT}_r^i(\mathbb{Z}), \mathrm{UT}_r(\mathbb{Z})] = \mathrm{UT}_r^{i+1}(\mathbb{Z}) \ \text{ and } \ \mathsf{Z}_{i-1}/\mathsf{Z}_i \simeq \mathbb{Z}^{r-i}$$

(see Example 1.4.10).

Hence, if $F \in \mathcal{E}(\mathcal{M})$, by linearization we associate to F the endomorphisms: $\Phi_{A_i} \in \mathcal{E}(\mathbf{T}^{n_i})$ with $n_i = r - i$ for $i = 1, \ldots, r - 1$. Then $n' = \min\{n_1, \ldots, n_l\} = 1$ and, by the previous theorem,

$$\bigcap\{\mathcal{P}(F) \ : \ F \in \mathcal{E}(\mathcal{M})\} \supset \bigcap\{\mathcal{P}(F') \ : \ F' \in \mathcal{E}(\mathbf{T}^1)\} = \mathbb{N}^* \setminus \{2\}.$$

Note that, for $r = 3$, the expanding endomorphism F of $\mathcal{M}$, which is defined in the linearization of $\mathrm{UT}_3(\mathbb{Z})$ by the integer matrices $A_1 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ and $A_2 = \det(A_1) = -2$, does not have periodic points of period 2:

$$p_F(2) = \mathcal{N}(F^2) - \mathcal{N}(F) = |\det(A_1^2 - \mathbb{I})||A_2^2 - 1| - |\det(A_1 - \mathbb{I})||A_2 - 1| = 3 - 3 = 0.$$

Q.E.D.

4

Commuting expanding maps on nil-manifolds

We turn now our attention to the study of the common fixed points of two given maps F and G which are expanding on a nil-manifold $\mathcal{M} = \mathsf{N}/\mathsf{Z}$.

We assume that it is possible to find two liftings $\widetilde{F}$ and $\widetilde{G}$ such that

$$\widetilde{F}^\sharp \circ \widetilde{G}^\sharp = \widetilde{G}^\sharp \circ \widetilde{F}^\sharp.$$

Note that this condition is necessary for F and G to have a common fixed point. Moreover, as we will see, it becomes sufficient if we assume that the cardinalities of the fixed points set of F and G, respectively $\mathcal{N}(F)$ and $\mathcal{N}(G)$, are relatively prime. In this case, F and G have one and only one common fixed point. This result will be attained in two steps. In the first one, we simplify the problem by choosing a suitable homeomorphism α_0 of $\mathcal{M}$ such that it conjugates both F and G to two endomorphisms:

$$\begin{cases} \alpha_0 \circ F \circ \alpha_0^{-1} = \Phi_A \\ \alpha_0 \circ G \circ \alpha_0^{-1} = R_\theta \circ \Phi_B \end{cases}$$

where θ belongs to the center of N and R_θ is the map induced on $\mathcal{M}$ by the translation $(\mathbb{I}, \theta) \in \mathrm{Aff}(\mathsf{N})$. In the second step, we solve first the torus case, and, then we generalize the argument to any nil-manifold, by exploiting the linearization of Z.

In the last section of this Chapter, as an application of the above discussion, we recover a result proved by A. S. A. Johnson and D. J. Rudolph in [JoRu], whereby, if $F, G \in \mathcal{E}^1(\mathbf{T}^1) \cap C^2(\mathbf{T}^1)$ are commuting, orientation-preserving maps such that the integers A and B, respectively the degree of F and G, do not have a common power then the above homeomorphism α_0 is actually a diffeomorphism.

§4.1 A common conjugation.

Let $F, G \in \mathcal{E}(\mathcal{M})$ where $\mathcal{M}$ is a closed manifold and take two liftings $\widetilde{F}$ and $\widetilde{G}$ respectively of F and G. If F and G commute, then $\widetilde{F} \circ \widetilde{G}$ and $\widetilde{G} \circ \widetilde{F}$ are two liftings of the same map. Thus there exists a $\gamma_0 \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$ such that

$$\widetilde{F} \circ \widetilde{G} = \gamma_0 \circ \widetilde{G} \circ \widetilde{F}. \tag{1}$$

Lemma 4.1.1 *The following identity holds:*

$$\widetilde{F}^\sharp \circ \widetilde{G}^\sharp = \gamma_0^\sharp \circ \widetilde{G}^\sharp \circ \widetilde{F}^\sharp \tag{2}$$

Proof. In fact, if $\gamma \in \mathcal{D}(\widetilde{\mathcal{M}}/\mathcal{M})$, then

$$\begin{cases} \widetilde{F} \circ \widetilde{G} \circ \gamma = \widetilde{F} \circ \widetilde{G}^\sharp(\gamma) \circ \widetilde{G} = (\widetilde{F}^\sharp \circ \widetilde{G}^\sharp)(\gamma) \circ \widetilde{F} \circ \widetilde{G} \\ \widetilde{G} \circ \widetilde{F} \circ \gamma = \widetilde{G} \circ \widetilde{F}^\sharp(\gamma) \circ \widetilde{F} = (\widetilde{G}^\sharp \circ \widetilde{F}^\sharp)(\gamma) \circ \widetilde{G} \circ \widetilde{F}, \end{cases}$$

and, by (1),

$$(\widetilde{F}^\sharp \circ \widetilde{G}^\sharp)(\gamma) \circ \widetilde{F} \circ \widetilde{G} = \gamma_0 \circ (\widetilde{G}^\sharp \circ \widetilde{F}^\sharp)(\gamma) \circ \gamma_0^{-1} \circ \widetilde{F} \circ \widetilde{G}.$$

This implies (2), because $\gamma_0^\sharp(\cdot) = \gamma_0 \circ (\cdot) \circ \gamma_0^{-1}$ and $\widetilde{F}$, $\widetilde{G}$ are homeomorphisms. Q.E.D.

Now, assume that $\mathcal{M} = \mathsf{N}/\mathsf{Z}$ is a nil-manifold. Denote by $C(\mathsf{N})$ the *center* of the group N, i. e. the subgroup of N whose elements commute with all elements of N.

Remark 4.1.2 Let $\Phi_{(A,a)}$ and $\Phi_{(B,b)}$ be endomorphisms associated respectively to F and G, that is, $(A,a)^\sharp = \widetilde{F}^\sharp$ and $(B,b)^\sharp = \widetilde{G}^\sharp$ for some liftings $\widetilde{F}$ and $\widetilde{G}$. Note that we can assume that $a = b = e$, the identity element of N, because

$$(A,a)^\sharp(\mathbb{I},p) = (A,a)(\mathbb{I},p)(A^{-1}, A^{-1}(a^{-1})) = (\mathbb{I}, aA(p)a^{-1}) \quad \forall p \in \mathsf{Z}.$$

The homomorphism $aA(\cdot)a^{-1}$ of Z is injective, and thus, by (c) of Theorem 1.4.7, it can be extended uniquely to an automorphism A' of N. Therefore $(A,a)^\sharp = (A',e)^\sharp$ and we can replace (A,a) by (A',e). The same can be done for (B,b).

Theorem 4.1.3 *If $F, G \in \mathcal{E}(\mathcal{M})$ commute and $\widetilde{F}^\sharp \circ \widetilde{G}^\sharp = \widetilde{G}^\sharp \circ \widetilde{F}^\sharp$, then there exists a homeomorphism α_0 of $\mathcal{M}$ such that $\tilde{\alpha}_0^\sharp = \mathrm{Id}$ and an element $\theta \in C(\mathsf{N})$ such that*

$$\begin{cases} \alpha_0 \circ F \circ \alpha_0^{-1} = \Phi_A \\ \alpha_0 \circ G \circ \alpha_0^{-1} = R_\theta \circ \Phi_B \end{cases} \tag{3}$$

where R_θ is the map induced on $\mathcal{M}$ by the translation $(\mathbb{I}, \theta) \in \mathrm{Aff}(\mathsf{N})$.

Proof. Since $\widetilde{F}^\sharp \circ \widetilde{G}^\sharp = \widetilde{G}^\sharp \circ \widetilde{F}^\sharp$, then, by Lemma 4.1.1, $\gamma_0 = (\mathbb{I}, r)$ is such that $\gamma_0^\sharp = \mathrm{Id}$. By (c) of Theorem 1.4.7, $\gamma_0^\sharp$ can be extended uniquely to the identity homomorphism of N; hence

$$\gamma_0^\sharp((\mathbb{I}, y)) = (\mathbb{I}, r)(\mathbb{I}, y)(\mathbb{I}, r)^{-1} = (\mathbb{I}, ryr^{-1}) = (\mathbb{I}, y) \quad \forall y \in \mathsf{N},$$

that is $r \in C(\mathsf{N})$. Moreover, $(A,e)^\sharp(B,e)^\sharp = (B,e)^\sharp(A,e)^\sharp$ implies that $AB = BA$.

As in Theorem 1.3.6, we denote by $\mathcal{S}_{\mathrm{Id}}$ the set of all continuous maps $\tilde{\alpha} : \mathsf{N} \to \mathsf{N}$ such that $\tilde{\alpha}^\sharp = \mathrm{Id}$, and define the following two maps

$$\mathcal{F}(\tilde{\alpha}) \stackrel{\mathrm{def}}{=} (A,e)^{-1} \circ \tilde{\alpha} \circ \widetilde{F} \quad \text{and} \quad \mathcal{G}_\theta(\tilde{\alpha}) \stackrel{\mathrm{def}}{=} (B,e)^{-1} \circ (\mathbb{I}, \theta) \circ \tilde{\alpha} \circ \widetilde{G},$$

where the parameter $\theta \in C(\mathsf{N})$ will be chosen in a suitable way later. The map $\mathcal{F}$ send $\mathcal{S}_{\mathrm{Id}}$ into itself, and, since $(\mathbb{I}, \theta) \in \mathcal{S}_{\mathrm{Id}}$, also $\mathcal{G}_\theta(\mathcal{S}_{\mathrm{Id}}) \subset \mathcal{S}_{\mathrm{Id}}$. Moreover, once again by Theorem 1.3.6, we know that $\mathcal{F}$ has exactly one fixed point $\tilde{\alpha}_0$ in $\mathcal{S}_{\mathrm{Id}}$ and $\tilde{\alpha}_0$ induces a homeomorphism α_0 on $\mathcal{M}$.

Now, it will be shown that the parameter θ can be chosen in such a way that the maps $\mathcal{G}_\theta$ and $\mathcal{F}$ commute and thus have the same fixed point $\tilde{\alpha}_0$.

First of all, for all $\tilde{\alpha} \in \mathcal{S}_{\mathrm{Id}}$,

$$\begin{aligned} (\mathcal{F} \circ \mathcal{G}_\theta)(\tilde{\alpha}) &= (A^{-1}, e) \circ \mathcal{G}_\theta(\tilde{\alpha}) \circ \widetilde{F} \\ &= (A^{-1}B^{-1}, A^{-1}B^{-1}(\theta)) \circ \tilde{\alpha} \circ \widetilde{G} \circ \widetilde{F}. \end{aligned}$$

Similarly, since $\gamma_0 = (\mathbb{I}, r)$ and $\tilde{\alpha}^\sharp = \mathrm{Id}$,

$$\begin{aligned}(\mathcal{G}_\theta \circ \mathcal{F})(\tilde{\alpha}) &= (B^{-1}, B^{-1}(\theta)) \circ \mathcal{F}(\tilde{\alpha}) \circ \widetilde{G} \\ &= (B^{-1}A^{-1}, B^{-1}(\theta)) \circ \tilde{\alpha} \circ \widetilde{F} \circ \widetilde{G} \\ &= (B^{-1}A^{-1}, B^{-1}(\theta A^{-1}(r))) \circ \tilde{\alpha} \circ \widetilde{G} \circ \widetilde{F}.\end{aligned}$$

Thus, $\mathcal{F}$ and $\mathcal{G}_\theta$ commute iff

$$(A^{-1}B^{-1}, A^{-1}B^{-1}(\theta)) = (B^{-1}A^{-1}, B^{-1}(\theta A^{-1}(r))).$$

Since $AB = BA$, then the preceding equation holds iff

$$A^{-1}B^{-1}(\theta) = B^{-1}(\theta A^{-1}(r)),$$

that is $(A, r)(\theta) = \theta$, because $r \in C(\mathsf{N})$. Note that θ is the only one fixed point of (A, r), because (A, r) is an expanding diffeomorphism of N.

Now, we show that $\theta \in C(\mathsf{N})$. Since the map $(A, r)^{-1}$ is a contraction in N, the sequence whose elements are $\theta_k \stackrel{\text{def}}{=} (A, r)^{-k}(r)$ for $k \geq 0$ converges to the fixed point θ. The first element is $\theta_0 = r \in C(\mathsf{N})$. If $\theta_k \in C(\mathsf{N})$, then, for all $y \in \mathsf{N}$,

$$\theta_{k+1}y = A^{-1}(r^{-1}\theta_k)y = A^{-1}(r^{-1}\theta_k A(y)) = A^{-1}(A(y)r^{-1}\theta_k) = y\theta_{k+1}.$$

Therefore, since by induction all the sequence stays in $C(\mathsf{N})$, and $C(\mathsf{N})$ is closed, the same holds for the limit θ. Q.E.D.

§4.2 A common fixed point theorem.

As a consequence of Theorem 4.1.3, we are able to establish a sufficient condition for the existence of a common fixed point for two commuting expanding maps depending only on the cardinalities of their fixed points sets.

The following property of the integer matrices will be useful later.

Definition 4.2.1 Let $X, Y \in \mathbb{Z}^{n\times n}$ be two integer matrices, not both equal to 0. A common left divisor $D \in \mathbb{Z}^{n\times n}$ of X and Y such that any common left divisor of X and Y is also left divisor of D will be called a *left greatest common divisor* of X and Y.

The following theorem holds.

Theorem 4.2.2 [Hu] *If $X, Y \in \mathbb{Z}^{n\times n}$ are two integer matrices, not both equal to 0, then, there exists a left greatest common divisor D of X and Y.*

Moreover there exist $P, Q \in \mathbb{Z}^{n\times n}$ such that

$$XQ + YP = D.$$

First we establish the result for the endomorphisms on the n-torus.

Lemma 4.2.3 *Let $\Phi_{(A,a)}, \Phi_{(B,b)} \in \mathcal{E}^1(\mathbf{T}^n)$ be two commuting endomorphisms. If*

$$\gcd(\mathcal{N}(\Phi_{(A,a)}), \mathcal{N}(\Phi_{(B,b)})) = 1$$

then they have one and only one common fixed point.

Proof. Since $\Phi_{(A,a)}, \Phi_{(B,b)}$ commute, then

$$(A,a)(B,b) = (\mathbb{I}, r)(B,b)(A,a) \text{ for some } r \in \mathbb{Z}^n.$$

This implies that $AB = BA$ and $r = (A - \mathbb{I})b - (B - \mathbb{I})a$.

We begin by establishing the existence. The point $x = \pi(y)$ is a common fixed point iff there exist $p, q \in \mathbb{Z}^n$ such that

$$\begin{cases} (A,a)(y) = (\mathbb{I}, p)(y) \\ (B,b)(y) = (\mathbb{I}, q)(y) \end{cases}.$$

Then, after having eliminated y, we find the equation which has to be solved with respect to $p, q \in \mathbb{Z}^n$

$$(A - \mathbb{I})q - (B - \mathbb{I})p = (A - \mathbb{I})b - (B - \mathbb{I})a = r. \tag{4}$$

By the above theorem, the matrices $(A-\mathbb{I}),(B-\mathbb{I})\in\mathbb{Z}^{n\times n}$ have a left greatest common divisor $D\in\mathbb{Z}^{n\times n}$ and there exist $P,Q\in\mathbb{Z}^{n\times n}$ such that

$$(A-\mathbb{I})Q-(B-\mathbb{I})P=D. \tag{5}$$

Since D is a common left divisor, $\det(D)\in\mathbb{Z}$ divides $|\det(A-\mathbb{I})|=\mathcal{N}(\Phi_{(A,a)})\geq 1$ and $|\det(B-\mathbb{I})|=\mathcal{N}(\Phi_{(B,b)})\geq 1$. But, by hypothesis, $\mathcal{N}(\Phi_{(A,a)}),\mathcal{N}(\Phi_{(B,b)})$ are relatively prime. Hence $\det(D)=\pm 1$ i.e., $D\in\mathrm{GL}(n,\mathbb{Z})$.

Multiplying both sides of (5) by $D^{-1}r\in\mathbb{Z}^n$ on the right, we can write

$$(A-\mathbb{I})Q(D^{-1}r)-(B-\mathbb{I})P(D^{-1}r)=D(D^{-1}r)=r$$

and (4) is solved by $p=P(D^{-1}r)\in\mathbb{Z}^n$ and $q=Q(D^{-1}r)\in\mathbb{Z}^n$.

As for uniqueness, suppose that $x_1,x_2\in\mathbf{T}^n$ are two common fixed points. If $\pi(y_i)=x_i$ for $i=1,2$ then

$$\left\{\begin{array}{l}Ay_1+a=y_1+p_1\\ By_1+b=y_1+q_1\end{array}\right. \quad\text{and}\quad \left\{\begin{array}{l}Ay_2+a=y_2+p_2\\ By_2+b=y_2+q_2\end{array}\right.$$

for some $p_1,p_2,q_1,q_2\in\mathbb{Z}^n$. Hence, $x_1x_2^{-1}=\pi(y_1-y_2)$ is a common fixed point of the homomorphisms Φ_A,Φ_B.

Both sets $\mathrm{Fix}_{\mathbf{T}^n}(\Phi_A)$ and $\mathrm{Fix}_{\mathbf{T}^n}(\Phi_B)$ are subgroups of $\mathbf{T}^n$. Therefore the order of the subgroup $\mathrm{Fix}_{\mathbf{T}^n}(\Phi_A)\cap\mathrm{Fix}_{\mathbf{T}^n}(\Phi_B)$ divides $\mathcal{N}(\Phi_A)=\mathcal{N}(\Phi_{(A,a)})$, and $\mathcal{N}(\Phi_B)=\mathcal{N}(\Phi_{(B,b)})$. Since, by hypothesis, these two numbers are relatively prime, then $\mathrm{Fix}_{\mathbf{T}^n}(\Phi_A)\cap\mathrm{Fix}_{\mathbf{T}^n}(\Phi_B)=\{1\}$ where 1 is the identity element of $\mathbf{T}^n$ and $x_1=x_2$. Q.E.D.

Now, we are ready to prove the main theorem for a nil-manifold $\mathcal{M}$.

Theorem 4.2.4 *Let $F,G\in\mathcal{E}(\mathcal{M})$ be two commuting maps. The following two fact holds:*

(a) if F and G have a common fixed point, then there are two liftings $\widetilde{F}$, $\widetilde{G}$ such that $\widetilde{F}^\sharp\circ\widetilde{G}^\sharp=\widetilde{G}^\sharp\circ\widetilde{F}^\sharp$, and there is a homeomorphism α_0 of $\mathcal{M}$ such

that

$$\begin{cases} \alpha_0 \circ F \circ \alpha_0^{-1} = \Phi_A \\ \alpha_0 \circ G \circ \alpha_0^{-1} = \Phi_B \end{cases};$$

(b) if $\widetilde{F}^\sharp \circ \widetilde{G}^\sharp = \widetilde{G}^\sharp \circ \widetilde{F}^\sharp$ *for some two liftings* $\widetilde{F}$ *and* $\widetilde{G}$ *and*

$$\gcd(\mathcal{N}(F), \mathcal{N}(G)) = 1,$$

then F *and* G *have one and only one common fixed point.*

Proof. (a) Let x_0 be a common fixed point and of F and G and let $y_0 \in \pi^{-1}(x_0)$. Then there exist $p, q \in \mathsf{Z}$ such that

$$\widetilde{F}(y_0) = (\mathbb{I}, p)y_0 \text{ and } \widetilde{G}(y_0) = (\mathbb{I}, q)y_0.$$

Now, replace the liftings $\widetilde{F}, \widetilde{G}$ by the liftings $\widetilde{F}_1 = (\mathbb{I}, p)^{-1}\widetilde{F}$ and $\widetilde{G}_1 = (\mathbb{I}, q)^{-1}\widetilde{G}$, then y_0 is their unique fixed point in N. By (1),

$$y_0 = (\widetilde{F} \circ \widetilde{G})(y_0) = (\gamma_0 \circ \widetilde{G} \circ \widetilde{F})(y_0) = \gamma(y_0).$$

Therefore, $\gamma_0 = (\mathbb{I}, e)$, and in Theorem 4.1.3 we can choose $\theta = e$.

(b) By Theorem 4.1.3, $F, G \in \mathcal{E}(\mathcal{M})$ can be conjugated to two commuting endomorphisms $\Phi_A, \Phi_{(B,\theta)} \in \mathcal{E}(\mathcal{M})$ with $\theta \in C(\mathsf{N})$ and $(A, e)(B, \theta) = (\mathbb{I}, r)(B, \theta)(A, e)$ for some $r \in C(\mathsf{N}) \cap \mathsf{Z}$.

Therefore, it is enough to prove the theorem when F, G are two expanding endomorphisms $\Phi_{(A,a)}, \Phi_{(B,b)}$ with $(A, a)(B, b) = (\mathbb{I}, r)(B, b)(A, a)$ and $r \in C(\mathsf{N}) \cap \mathsf{Z}$. This will be established by induction on the nilpotency class of N.

When the nilpotency class is equal to 1, then N is abelian. Hence $\mathcal{M}$ is homeomorphic to a n-torus and, by the preceding lemma, there is one and only one common fixed point.

Now, assume that the nilpotency class of N is $l \geq 2$. Let $\mathsf{N}_1 = [\mathsf{N}, \mathsf{N}]$ and $\mathsf{Z}_1 \stackrel{\text{def}}{=} \mathsf{Z} \cap \mathsf{N}_1$ and define the two nil-manifolds

$$\mathcal{M}' = (\mathsf{N}/\mathsf{N}_1)/(\mathsf{Z}/\mathsf{Z}_1) \text{ and } \mathcal{M}'' = \mathsf{N}_1/\mathsf{Z}_1$$

where the nilpotency classes of N/N_1 and N_1 are respectively equal to 1 and $l-1$.

Then, by Theorem 3.6.1,

$$\mathcal{N}(\Phi_{(A,a)}) = \mathcal{N}(\Phi'_{(A,a)})\mathcal{N}(\Phi''_{(A,a)}) \text{ and } \mathcal{N}(\Phi_{(B,b)}) = \mathcal{N}(\Phi'_{(B,b)})\mathcal{N}(\Phi''_{(B,b)})$$

where Φ' and Φ'' denote the endomorphisms induced by Φ respectively on $\mathcal{M}'$ and $\mathcal{M}''$. Therefore, $\gcd(\mathcal{N}(\Phi'_{(A,a)}, \mathcal{N}(\Phi'_{(B,b)})) = 1$.

Since the nilpotency class of N/N_1 is equal to 1, then $\mathcal{M}'$ is homeomorphic to a torus and, by the preceding lemma, there is $y \in \mathsf{N}$ such that

$$\begin{cases} aA(y\mathsf{N}_1\mathsf{Z}/\mathsf{Z}_1) = y\mathsf{N}_1\mathsf{Z}/\mathsf{Z}_1 \\ bB(y\mathsf{N}_1\mathsf{Z}/\mathsf{Z}_1) = y\mathsf{N}_1\mathsf{Z}/\mathsf{Z}_1. \end{cases} \tag{6}$$

Let $a_1 \overset{\text{def}}{=} y^{-1}aA(y)$ and $b_1 \overset{\text{def}}{=} y^{-1}bB(y)$, then $\Phi''_{(A,a_1)}, \Phi''_{(B,b_1)} \in \mathcal{E}(\mathcal{M}'')$ and

$$(A, a_1)(B, b_1) = (\mathbb{I}, r)(B, b_1)(A, a_1)$$

with $r \in C(\mathsf{N}) \subset C(\mathsf{N}_1)$. Moreover,

$$\gcd(\mathcal{N}(\Phi''_{(A,a_1)}, \mathcal{N}(\Phi''_{(B,b_1)})) = \gcd(\mathcal{N}(\Phi''_{(A,a)}, \mathcal{N}(\Phi''_{(B,b)})) = 1.$$

Hence, since the nilpotency class of N_1 is less than l, by the inductive hypothesis, there is $y_1 \in \mathsf{N}_1$ such that

$$\begin{cases} a_1A(y_1\mathsf{Z}_1) = y_1\mathsf{Z}_1 \\ b_1B(y_1\mathsf{Z}_1) = y_1\mathsf{Z}_1. \end{cases} \tag{7}$$

Therefore, equations (6) and (7) yield

$$\begin{cases} aA(yy_1\mathsf{Z}) = yy_1\mathsf{Z} \\ bB(yy_1\mathsf{Z}) = yy_1\mathsf{Z} \end{cases}$$

that is, $\pi(yy_1) \in \mathcal{M}$ is the unique common fixed point of the maps $\Phi_{(A,a)}$ and $\Phi_{(B,b)}$. Q.E.D.

Remark 4.2.5 Note that the condition (b) of the previous theorem is not necessary. In fact, let $\mathcal{M} = \mathbf{T}^2$ and define the integer matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then, the commuting endomorphism Φ_A and Φ_B are expanding on $\mathbf{T}^2$ and $\mathcal{N}(\Phi_A) = \mathcal{N}(\Phi_B) = 2$, whereas 1 is their unique common fixed point.

§4.3 Commuting maps on $\mathbf{T}^1$.

As a application of the previous facts, we recover, a result established by A. S. A. Johnson and D. J. Rudolph in [JoRu].

Definition 4.3.1 Let A, B be two integers greater than 1. We denote by $<A, B>$ the multiplicative semigroup in $\mathbb{N}$ generated by A and B. We will say that $<A, B>$ is *nonlacunary* if $<A, B>$ is not contained in a semigroup generated by only one integer, or, in a equivalent way, if A and B do not have a common power (i. e. $A^s \neq B^t$ for all $s, t \geq 1$).

For example, the semigroup $<6, 10>$ is nonlacunary, whereas $<9, 27>$ is not, because it is contained in the singly generated semigroup $<3>$.

Theorem 4.3.2 [Jo] *Let $A, B \geq 2$ be such that $< A, B >$ is a nonlacunary semigroup. If μ is a Borel normalized measure of $\mathbf{T}^1$, invariant and ergodic for Φ_A and Φ_B, then either μ is equal to the normalized Lebesgue measure ν or $h_\mu(\Phi_A) = h_\mu(\Phi_B) = 0$*

Here is the main result.

Theorem 4.3.3 [JoRu] *Let $F, G \in \mathcal{E}^1(\mathbf{T}^1) \cap C^2(\mathbf{T}^1)$ be two commuting, orientation preserving maps with A and B respectively the degree of F and G. If $< A, B >$ is a nonlacunary semigroup then the map α_0 in (3) is a diffeomorphism.*

Proof. By Corollary 2.2.7, since F and G commute, there exists a normalized measure μ equivalent to the Lebesgue measure ν, which is invariant and ergodic for both F and G, and $\frac{d\mu}{d\nu} = h$ is a continuous map.

Moreover, if $x_0 \in \mathrm{Fix}_{\mathbf{T}^1}(F)$ then, for $k \geq 1$,

$$G^k(x_0) = G^k(F(x_0)) = F(G^k(x_0)) \in \mathrm{Fix}_{\mathbf{T}^1}(F)$$

and, since F has a finite number of fixed points, then there is an integer m such that F and G^m have a common fixed point on $\mathbf{T}^1$. Note that $G^m \in \mathcal{E}^1(\mathbf{T}^1)$ and its degree is equal to $B^m \geq 2$.

Therefore, since $\mathbf{T}^1 = \mathbb{R}/\mathbb{Z}$ is a nil-manifold, by (a) of Theorem 4.2.4, there exists a homeomorphism α_0 such that

$$\begin{cases} \alpha_0 \circ F \circ \alpha_0^{-1} = \Phi_A \\ \alpha_0 \circ G^m \circ \alpha_0^{-1} = \Phi_{B^m} \end{cases} .$$

Now, define the normalized measure $\tilde{\mu} \stackrel{\mathrm{def}}{=} \mu \circ \alpha_0^{-1}$. Then $\tilde{\mu}$ is invariant and ergodic for both Φ_A and Φ_{B^m}. Moreover, as regards the entropies, by Remark 2.2.5:

$$h_{\tilde{\mu}}(\Phi_A) = h_\mu(F) > 0 \ \text{ and } \ h_{\tilde{\mu}}(\Phi_{B^m}) = h_\mu(G^m) > 0.$$

Since also the semigroup $< A, B^m >$ is nonlacunary, the above facts implies, by the previous theorem, that $\tilde{\mu}$ is the normalized Lebesgue measure ν. So,

$$\alpha_0(x) = \mu([x_0, x]) = \int_{x_0}^{x} h(s) d\nu(s) \qquad \forall x \in \mathbf{T}^1$$

and therefore $\alpha_0 \in C^1(\mathbf{T}^1)$. Q.E.D.

5

Extensions of differentiably expanding maps on $\mathbf{T}^n$

In this Chapter, given a differentiably expanding map F of $\mathbf{T}^n$, we will investigate the fixed point set of an extension $\widehat{F} : \overline{\Delta}^n \to \overline{\Delta}^n$ where $\Delta^n \stackrel{\text{def}}{=} \{z \in \mathbb{C}^n \ : \ |z| < 1\}$ is the n-dimensional open polydisc.

First, we consider the case where $n = 1$ and $\widehat{F}$ is a C^1-map. If Φ_A is the endomorphism associated, then the integer A is actually the number of windings around $\mathbf{T}^1$ of the path $F(e^{2\pi it})$ with $t \in [0, 1]$; a winding is counted positively if it is oriented counterclockwise and negatively in the opposite case. We will prove that, if F preserves the orientation, i. e. $A \geq 2$, then every C^1-extension $\widehat{F}$ has a fixed point inside Δ. On the contrary, if $A \leq -2$, there exists a C^1-estension $\widehat{F}$ that has no fixed point in Δ. Moreover, by two counterexamples, we show that we can not weaken neither the hypothesis F is differentiably expanding by F is expanding nor the regularity of the extensions from C^1 to continuous.

Next, we study the holomorphic extensions for $n \geq 1$: more precisely, the maps belonging to the set

$$\mathcal{I}(\Delta^n) \stackrel{\text{def}}{=} \mathrm{Hol}(\Delta^n, \Delta^n) \cap C(\overline{\Delta}^n, \overline{\Delta}^n) \cap C(\mathbf{T}^n, \mathbf{T}^n)$$

whose restrictions to $\mathbf{T}^n$ are differentiably expanding. For $n = 1$, we recover a result proved by N. F. G. Martin in [Ma]: if $F \in \mathcal{I}(\Delta^1)$, $F_{|\mathbf{T}^1}$ is differentiably expanding iff F is non-invertible and has a unique fixed point in Δ. For $n \geq 2$ we generalize this result in the following form: if $F \in \mathcal{I}(\Delta^n)$ and $F_{|\mathbf{T}^n} \in \mathcal{E}^1(\mathbf{T}^n)$, then F is non-invertible and has a unique fixed point in Δ. However, we will show that the opposite implication is false. The Chapter ends with a discussion about the fixed points contained in the boundary of the n-polydisc for a map $F \in \mathcal{I}(\Delta^n)$ such that $F_{|\mathbf{T}^n} \in \mathcal{E}^1(\mathbf{T}^n)$. Rearranging some results of the previous Chapters, we prove that: if C is a *boundary component* of the polydisc Δ^n, (i. e. the cartesian product in some order of $r \geq 1$ copies of $\mathbf{T}^1$ and $s \geq 0$ copies of Δ with $r+s=n$), and if $F(C) \cap C \neq \emptyset$, then

$$F(C) \subset C \;\text{ and }\; 1 \leq \operatorname{card}(\operatorname{Fix}_C(F)) \leq \mathcal{N}(F),$$

and therefore $1 + \mathcal{N}(F_{|\mathbf{T}^n}) \leq \operatorname{card}(\operatorname{Fix}_{\overline{\Delta}^n}(F)) \leq 1 + (2^n - 1)\mathcal{N}(F_{|\mathbf{T}^n})$. Moreover, also the case of two commuting maps is investigated.

§5.1 C^1-extensions on $\mathbf{T}^1$.

We begin by recalling the definition of the fixed point index on $\overline{\Delta}$ where

$$\Delta \stackrel{\text{def}}{=} \{z \in \mathbb{C} \;:\; |z| < 1\}.$$

Let $r_{\overline{\Delta}} : \mathbb{C} \to \overline{\Delta}$ be the following retraction of $\mathbb{C}$ on $\overline{\Delta}$:

$$r_{\overline{\Delta}}(z) \stackrel{\text{def}}{=} \begin{cases} z & \text{if } z \in \overline{\Delta} \\ \frac{z}{|z|} & \text{if } z \in \mathbb{C}/\overline{\Delta} \end{cases}.$$

Let $F \in \mathcal{C}_U(\overline{\Delta})$ with U open subset of $\overline{\Delta}$ containing only one fixed point x of F. We define the index $i(\overline{\Delta}, F, U) = i(\overline{\Delta}, F, x)$ as the degree of the map

$$\varphi(z) \stackrel{\text{def}}{=} \frac{z - (F \circ r_{\overline{\Delta}})(z)}{|z - (F \circ r_{\overline{\Delta}})(z)|} \quad \text{for } z \in S_\epsilon$$

where $S_\epsilon \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z - x| = \epsilon\}$ with $\epsilon > 0$ suitably small such that $S_\epsilon \cap \overline{\Delta} \subset U$. It can be verified that this index satisfies the axioms listed at the beginning of Chapter 3 (see [Ji] p. 11).

Definition 5.1.1 We denote by $E^1(F)$ the set of all C^1-extensions of F inside $\overline{\Delta}$:

$$E^1(F) \stackrel{\text{def}}{=} \{\widehat{F} \in C^1(\overline{\Delta}, \overline{\Delta}) \;:\; \widehat{F}_{|\mathbf{T}^1} \equiv F\}.$$

The results established in the following two theorems will turn out to be fundamental tools later. The first one gives a restriction on the possible values of $i(\overline{\Delta}, \widehat{F}, x)$ when $\widehat{F} \in E^1(F)$ and $x \in \mathbf{T}^1$ is a fixed point of F. The second one shows that a map $F_1 \in C^1(\overline{\Delta}, \overline{\Delta})$ without fixed points on the boundary of a convex closed set $C \subset \overline{\Delta}$, can be modified to a map $F_2 \in C^1(\overline{\Delta}, \overline{\Delta})$ with at most one fixed point in the interior of C, without changing the original map out of $\mathrm{int}(C)$.

Theorem 5.1.2 [BGS] *Let $F \in C^1(\mathbf{T}^1, \mathbf{T}^1)$ and let $x \in \mathbf{T}^1$ be an isolated fixed point of F. If F is transversally fixed in x, i. e. the derivative $D_x F$ of F in x is different from 1, and $\widehat{F} \in E^1(F)$, then either $i(\overline{\Delta}, \widehat{F}, x) = 0$ or $i(\overline{\Delta}, \widehat{F}, x) = i(\mathbf{T}^1, F, x)$ which is either 1 or -1.*

Theorem 5.1.3 [BGS] *Let C be a closed convex set in $\overline{\Delta}$.*

If $F_1 \in C^1(\overline{\Delta}, \overline{\Delta})$ is a map without fixed points on ∂C, then there is a C^1-map $F_2 \in C^1(\overline{\Delta}, \overline{\Delta})$ such that:

(a) F_1 and F_2 coincide in $\overline{\Delta}/\mathrm{int}(C)$;

(b) if $i(\overline{\Delta}, F_1, \mathrm{int}(C)) \neq 0$, then F_2 has exactly one fixed point $y \in \mathrm{int}(C)$ and $i(\overline{\Delta}, F_2, y) = i(\overline{\Delta}, F_1, \mathrm{int}(C))$;

(c) if $i(\overline{\Delta}, F_1, \mathrm{int}(C)) = 0$, then F_2 does not have fixed points in $\mathrm{int}(C)$.

Remark 5.1.4 Since $\overline{\Delta}$ is simply connected, every continuous map $F : \overline{\Delta} \to \overline{\Delta}$ is homotopic to the constant map identically zero, and therefore, by the homotopy invariance of the index,

$$L(\overline{\Delta}, F) = L(\overline{\Delta}, 0) = i(\overline{\Delta}, 0, 0) = \deg(\varphi) = 1,$$

where φ is the map $z \to \frac{z}{|z|}$ restricted to a small circle about 0.

Now, we can establish the main theorem of this section which generalizes a similar result established in [BrGr]:

Theorem 5.1.5 [Ta1] *Let* $F \in \mathcal{E}^1(\mathbf{T}^1)$ *with associated endomorphism* Φ_A. *Then:*

(a) if $A \geq 2$, *every* $\widehat{F} \in E^1(F)$ *has a fixed point in* Δ;

(b) if $A \leq -2$, *there exists a map* $\widehat{F} \in E^1(F)$ *that has no fixed point in* Δ.

Proof. By Theorem 3.4.11, the map F has $\mathcal{N}(F) = \mathcal{N}(\Phi_A) = |A-1| \geq 1$ fixed points on $\mathbf{T}^1$ and, if x is one of them,

$$i(\mathbf{T}^1, F, x) = i(\mathbf{T}^1, \Phi_A, \alpha_0(x)) = \operatorname{sgn}(1-A) = \begin{cases} -1 & \text{if } A \geq 2 \\ 1 & \text{if } A \leq -2 \end{cases} \tag{1}$$

where α_0 is a homeomorphism that conjugates F to the endomorphism Φ_A. Clearly, since $F \in \mathcal{E}^1(\mathbf{T}^1)$, F is transversally fixed in each of its fixed point. Now we distinguish the two cases.

(a) If $A \geq 2$, we assume that there exists a map $\widehat{F} \in E^1(F)$ without any fixed point in Δ. If $x \in \mathbf{T}^1$ is a fixed point of F, then, by Theorem 5.1.2 and (1),

$$\text{either } \; i(\overline{\Delta}, \widehat{F}, x) = 0 \; \text{ or } \; i(\overline{\Delta}, \widehat{F}, x) = -1. \tag{2}$$

Therefore, by (2), the normalization property implies

$$1 = L(\overline{\Delta}, \widehat{F}) = \sum_{x \in \mathrm{Fix}_{\mathbf{T}^1}(F)} i(\overline{\Delta}, \widehat{F}, x) \leq 0 \,,$$

which is a contradiction.

(b) If $A \leq -2$, we extend F inside $\overline{\Delta}$ first in the following manner

$$\widehat{F}_0(z) \stackrel{\text{def}}{=} \begin{cases} |z|^2 F(\frac{z}{|z|}) & \text{if } z \in \overline{\Delta} \setminus \{0\} \\ 0 & \text{if } z = 0 \,. \end{cases}$$

It is easy to see that $\widehat{F}_0 \in E^1(F)$ and that 0 is the only fixed point of $\widehat{F}_0$ in Δ. Since $|z| > |\widehat{F}_0(z)|$ for $z \in \Delta/\{0\}$, then the local degree of the map

$$\varphi(z) \stackrel{\text{def}}{=} \frac{z - \widehat{F}_0(z)}{|z - \widehat{F}_0(z)|}$$

at 0, is equal to 1 and therefore $i(\overline{\Delta}, \widehat{F}_0, 0) = 1$.

Moreover, if $x \in \mathrm{Fix}_{\mathbf{T}^1}(F)$, by Theorem 5.1.2 and (1),

$$\text{either } \; i(\overline{\Delta}, \widehat{F}_0, x) = 0 \; \text{ or } \; i(\overline{\Delta}, \widehat{F}_0, x) = 1. \tag{3}$$

Since, by the normalization property,

$$\sum_{x \in \mathrm{Fix}_{\mathbf{T}^1}(F)} i(\overline{\Delta}, \widehat{F}_0, x) = L(\overline{\Delta}, \widehat{F}_0) - i(\overline{\Delta}, \widehat{F}_0, 0) = 1 - 1 = 0,$$

then, by (3), $i(\overline{\Delta}, \widehat{F}_0, x) = 0$ for all $x \in \mathrm{Fix}_{\mathbf{T}^1}(F)$.

Choose one of these fixed points, say $x_0 \in \mathbf{T}^1$, and let C_1 be a convex closed set in $\overline{\Delta}$ with $\mathrm{int}(C_1) \neq \emptyset$, such that $\overline{\Delta} \setminus C_1$ is the union of two disjoint connected open sets U, V with $0 \in U$ and $\mathrm{Fix}_{\mathbf{T}^1}(F) \cap V = \{x_0\}$ (for example, C_1 can be a thin linear strip suitably close to x_0, intersected with $\overline{\Delta}$).

Let $\varrho : \overline{\Delta} \to [0, 1]$ be a C^1-map such that $\varrho_{|U} \equiv 0$ and $\varrho_{|V} \equiv 1$ and define for $z \in \overline{\Delta}$,

$$\widehat{F}_1(z) \stackrel{\text{def}}{=} \begin{cases} [|z|^2 + (1 - |z|^2)\varrho(z)]F(\frac{z}{|z|}) & \text{if } z \in \overline{\Delta} \setminus \{0\} \\ 0 & \text{if } z = 0 \, . \end{cases}$$

Clearly, $\widehat{F}_1 \in E^1(F)$. Since $\widehat{F}_1(z) = F(\frac{z}{|z|})$ for $z \in V$ and, by hypothesis, $A \leq 2$, then

$$i(\overline{\Delta}, \widehat{F}_1, x_0) = i(\mathbf{T}^1, F, x_0) = 1.$$

Moreover, $\widehat{F}_1(z) = \widehat{F}_0(z)$ for $z \in U$; thus

$$i(\overline{\Delta}, \widehat{F}_1, 0) = i(\overline{\Delta}, \widehat{F}_0, 0) = 1.$$

Therefore,

$$i(\overline{\Delta}, \widehat{F}_1, \mathrm{int}(C_1)) = 1 - i(\overline{\Delta}, \widehat{F}_1, 0) - i(\overline{\Delta}, \widehat{F}_1, x_0) = -1$$

and $\widehat{F}_1$ has at least one fixed point in $\mathrm{int}(C_1)$. So, by Theorem 5.1.3, we can modify $\widehat{F}_1$ on C_1 to a map $\widehat{F}_2 \in E^1(F)$ such that $\widehat{F}_2$ has just one single fixed point $y \in \mathrm{int}(C_1)$, with $i(\overline{\Delta}, \widehat{F}_2, y) = -1$ and $i(\overline{\Delta}, \widehat{F}_2, 0) = i(\overline{\Delta}, \widehat{F}_1, 0) = 1$.

Let $C_2 = \{z \in \overline{\Delta} : |z| < \frac{|y|+1}{2}\}$. Then, C_2 contains the two fixed points, 0 and y, of $\widehat{F}_2$ in Δ and

$$i(\overline{\Delta}, \widehat{F}_2, \mathrm{int}(C_2)) = i(\overline{\Delta}, \widehat{F}_2, 0) + i(\overline{\Delta}, \widehat{F}_2, y) = 1 - 1 = 0.$$

Applying once more Theorem 5.1.3, we can modify $\widehat{F}_2$ on C_2 to a map $\widehat{F}_3 \in E^1(F)$ without fixed points in $\mathrm{int}(C_2)$ and therefore in Δ. Q.E.D.

Remark 5.1.6 Note that, in the previous theorem, when $A \geq 2$, we can not assume $F \in \mathcal{E}(\mathbf{T}^1)$ because the holomorphic map $\widehat{F} : \overline{\Delta} \to \overline{\Delta}$, defined by

$$\widehat{F}(z) \stackrel{\text{def}}{=} \frac{1+3z^2}{3+z^2} \qquad \forall z \in \overline{\Delta},$$

has no fixed point in Δ, whereas, by Remark 1.4.5, $F = \widehat{F}_{|\mathbf{T}^1} \in \mathcal{E}(\mathbf{T}^1)$ has one fixed point $z = 1$ with $D_1 F = 1$ and is topologically conjugate to the endomorphism $\Phi_{2I} \in \mathcal{E}^1(\mathbf{T}^1)$.

Nevertheless, we can not weaken the hypothesis on the regularity of the extension $\widehat{F}$ of F because, if $F \in C(\mathbf{T}^1, \mathbf{T}^1)$ has a a fixed point $x_0 \in \mathbf{T}^1$, then there always exists an extension $\widehat{F} \in C(\overline{\Delta}, \overline{\Delta})$ which has no fixed points in Δ. The construction of $\widehat{F}$ is rather simple: for each $z \in \overline{\Delta} \setminus \{x_0\}$ the ray from x_0 through z intersects $\mathbf{T}^1$ in a point x and we may write $z = (1-r_z)x_0 + r_z x$ for some $r_z \in [0,1]$. If $\widehat{F}$ is the continuous map defined by

$$\widehat{F}(z) \stackrel{\text{def}}{=} \begin{cases} (1-r_z^2)x_0 + r_z^2 F(x) & \text{if } z \in \overline{\Delta} \setminus \{x_0\} \\ x_0 & \text{if } z = x_0, \end{cases}$$

then $\widehat{F}_{|\mathbf{T}^1} \equiv F$ and $\mathrm{Fix}_\Delta(\widehat{F}) = \emptyset$.

§5.2 Holomorphic extensions on $\mathbf{T}^1$.

Definition 5.2.1 A finite Blaschke product B is a function defined on $\overline{\Delta}$ by the equation:

$$B(z) \stackrel{\text{def}}{=} e^{i\theta} \prod_{j=1}^{N} \frac{z - a_j}{1 - \overline{a}_j z} \qquad \forall z \in \overline{\Delta}$$

where N is a positive integer, $\theta \in \mathbb{R}$ and $a_1 \dots a_N \in \Delta$.

Remark 5.2.2 Notice that $B_{|\mathbf{T}^1}$ is a N-to-1 map of $\mathbf{T}^1$ onto $\mathbf{T}^1$, which is determined, up to a rotation, by the zeros $a_1, \ldots, a_N$. When $N = 1$, B is the automorphism of $\overline{\Delta}$, $T_a(z) = \frac{z-a}{1-\overline{a}z}$ where $a \in \Delta$. Moreover, if a continuous surjective self-map of $\mathbf{T}^1$ has a holomorphic extension in Δ, then it can be realized as the restriction on $\mathbf{T}^1$ of a finite Blaschke product (see [Rud1] p. 310).

The modulus of the derivative of B in a point $x \in \mathbf{T}^1$ is given by

$$|B'(x)| = \sum_{j=1}^{N} \frac{1-|a_j|^2}{|x-a_j|^2}.$$

Now, we can characterize the Blaschke products which are differentiably expanding on $\mathbf{T}^1$.

Proposition 5.2.3 [Ma] *The following facts hold:*

(a) A finite Blaschke product B is differentiably expanding on $\mathbf{T}^1$ iff $N \geq 2$ and B has a unique fixed point $a \in \Delta$.

(b) If $B_{|\mathbf{T}^1} \in \mathcal{E}^1(\mathbf{T}^1)$, then the normalized measure ν_a such that

$$\frac{d\nu_a}{d\nu}(x) = \frac{1-|a|^2}{|x-a|^2} \qquad \forall x \in \mathbf{T}^1$$

is invariant under $B_{|\mathbf{T}^1}$, and

$$0 < h_{\nu_a}(B_{|\mathbf{T}^1}) = \int_{\mathbf{T}^1} \frac{1-|a|^2}{|x-a|^2} \log(\sum_{j=1}^{N} \frac{1-|a_j|^2}{|x-a_j|^2}) d\nu(x) \leq h_{\mathrm{top}}(B_{|\mathbf{T}^1}) = \log(N)$$

Moreover the equality $h_{\nu_a}(B_{|\mathbf{T}^1}) = h_{\mathrm{top}}(B_{|\mathbf{T}^1})$ holds iff B is conjugate, up to a rotation, to the map $z \to z^N$ by the automorphisms T_a.

Proof. (a) Assume that $B_{|\mathbf{T}^1} \in \mathcal{E}^1(\mathbf{T}^1)$. Then, by Theorem 1.2.7, $N \geq 2$. If B has no fixed point in Δ, then there is a $w \in \mathbf{T}^1$, the *Wolff point* (see for example [Bu]), such that $B(w) = w$ and $|B'(w)| \leq 1$. But, since $B_{|\mathbf{T}^1}$ is differentiably expanding, $|B'(w)| > 1$ and thus, we have a contradiction.

On the other hand, if $B(a) = a \in \Delta$ then the map $B_0 \stackrel{\text{def}}{=} T_a \circ B \circ T_a^{-1}$ is a new Blaschke product with zeros $a_1^{(0)}, \ldots, a_N^{(0)}$. Since $N \geq 2$ and $B_0(0) = 0$,

$$\lambda \stackrel{\text{def}}{=} \min_{x \in \mathbf{T}^1} \{|B_0'(x)|\} \geq \sum_{j=1}^{N} \frac{1 - |a_j^{(0)}|}{1 + |a_j^{(0)}|} > 1 \text{ and } c \stackrel{\text{def}}{=} \min_{x \in \mathbf{T}^1} \{|T_a(x)|\} \geq \frac{1 - |a|}{1 + |a|} > 0.$$

Therefore $|(B^k)'(x)| \geq c^2 \lambda^k$ for $k \geq 1$ and $x \in \mathbf{T}^1$. If $k_0 \geq 1$ is such that $c^2 \lambda^{k_0} > 1$, then $B_{|\mathbf{T}^1}^{k_0}$ is differentiably expanding and, by Proposition 1.4.6, also $B_{|\mathbf{T}^1} \in \mathcal{E}^1(\mathbf{T}^1)$.

(b) Let $\mathcal{P}[\cdot]$ be the one-dimensional *Poisson integral* defined as follows

$$\mathcal{P}[f](z) = \int_{\mathbf{T}^1} f(x) \frac{1 - |z|^2}{|x - z|^2} d\nu(x) \quad \forall f \in L^1(\mathbf{T}^1, \mathbb{C}) \text{ and } \forall z \in \Delta.$$

If $g \in C(\mathbf{T}^1, \mathbb{R})$, the boundary values of the harmonic maps $\mathcal{P}[g] \circ B$ and $\mathcal{P}[g \circ B]$ coincide on $\mathbf{T}^1$, and therefore they coincide also in Δ. Thus, for all $z \in \Delta$

$$\int_{\mathbf{T}^1} g(x) d\nu_{B(z)}(x) = \mathcal{P}[g](B(z)) = \mathcal{P}[g \circ B](z) = \int_{\mathbf{T}^1} g(B(x)) d\nu_z(x),$$

that is, $\nu_z \circ (B_{|\mathbf{T}^1})^{-1} = \nu_{B(z)}$.

Since the sequence of iterates $\{B^k(z)\}$ converges to the fixed point a for all $z \in \Delta$ (see [Bu]), then, by Theorem 2.2.4 (note that $\nu = \nu_0$),

$$\nu_0 \circ (B_{|\mathbf{T}^1})^{-k} = \nu_{B^k(0)} \xrightarrow{*} \nu_a$$

which is invariant by $B_{|\mathbf{T}^1}$. Moreover

$$h_{\nu_a}(B_{|\mathbf{T}^1}) = h_\nu(B_{0|\mathbf{T}^1}) \text{ and } h_{\text{top}}(B_{|\mathbf{T}^1}) = h_{\text{top}}(B_{0|\mathbf{T}^1}).$$

Thus, by Theorem 2.2.6, the equality holds iff $|B'(x)| = N$ for all $x \in \mathbf{T}^1$, that is, $B(z) = e^{i\theta} z^N$ for $z \in \overline{\Delta}$. Q.E.D.

§5.3 Holomorphic extensions on $\mathbf{T}^n$.

The following theorem, proved by J. P. Vigué , gives some properties of the fixed point set of a holomorphic self-map of a bounded convex domain $D \subset \mathbb{C}^n$.

Definition 5.3.1 A map $\varphi \in \mathrm{Hol}(\Delta, D)$ is said to be a *complex geodesic* if it is an isometry between the Carathéodory distance of Δ and D.

Theorem 5.3.2 [Vi] *Let $D \subset \mathbb{C}^n$ be a bounded convex domain and $F \in \mathrm{Hol}(D, D)$ then*

(a) if the set of fixed points of F, $\mathrm{Fix}_D(F)$, is not empty, then it is a holomorphic retract of D, i. e. there exists a map $\varrho \in \mathrm{Hol}(D, D)$ such that $\varrho^2 = \varrho$ and $\varrho(D) = \mathrm{Fix}_D(F)$;

(b) if z, w are two fixed points of F, there exists a complex geodesic φ such that

$$z, w \in \varphi(\Delta) \subset \mathrm{Fix}_D(F).$$

Remark 5.3.3 When $D = \Delta^n$, the Carathéodory distance of Δ^n is equal to

$$c_{\Delta^n}(z, w) = \max_{i=1,\dots,n} \{c_\Delta(z_i, w_i)\} \ \text{ with } \ c_\Delta(z_i, w_i) = \tanh^{-1}\Big(\Big|\frac{z_i - w_i}{1 - z_i w_i}\Big|\Big).$$

Therefore, a map $\varphi \in \mathrm{Hol}(\Delta, \Delta^n)$ is a complex geodesics of Δ^n iff

$$c_{\Delta^n}(\varphi(z), \varphi(w)) = \max_{i=1,\dots,n} \{c_\Delta(\varphi_i(z), \varphi_i(w))\} = c_\Delta(z, w) \quad \forall z, w \in \Delta,$$

hence, by the Schwarz lemma (see [Rud1] p. 254), iff at least one component φ_i of φ is an automorphism of Δ.

Definition 5.3.4 Let $\mathrm{Hol}(\Delta^n, \Delta^n)$ be the set of all holomorphic maps of Δ^n into Δ^n, and define the set of maps

$$\mathcal{I}(\Delta^n) \stackrel{\text{def}}{=} \mathrm{Hol}(\Delta^n, \Delta^n) \cap C(\overline{\Delta}^n, \overline{\Delta}^n) \cap C(\mathbf{T}^n, \mathbf{T}^n).$$

If $F = (f_1, \dots, f_n) \in \mathcal{I}(\Delta^n)$, every component map $f_i : \overline{\Delta}^n \to \overline{\Delta}$ is *inner* in Δ^n and continuous on $\overline{\Delta}^n$.

The following theorem characterizes this class of maps, and in particular shows that $\mathcal{I}(\Delta)$ consists of all finite Blaschke products.

Theorem 5.3.5 [Rud2] *If* $f : \overline{\Delta}^n \to \overline{\Delta}$ *is inner in* Δ^n *and continuous on* $\overline{\Delta}^n$*, then it is a rational function*[1]*:*

$$f(z) = \frac{M(z)\overline{Q}(\frac{1}{z})}{Q(z)} \tag{4}$$

where Q *is a polynomial in* $\mathbb{C}[z_1, \ldots, z_n]$ *which has no zero in* $\overline{\Delta}^n$*,* $\overline{Q}$ *is the polynomial* Q *with the complex conjugate coefficients.* M *is a monomial whose coefficient has modulus one and is such that* $P(z) \overset{\text{def}}{=} M(z)\overline{Q}(\frac{1}{z})$ *is a polynomial in* $\mathbb{C}[z_1, \ldots, z_n]$*.*

Remark 5.3.6 If $f \in L^1(\mathbf{T}^n, \mathbb{C})$, then the value of the n-dimensional *Poisson integral* $\mathcal{P}[f]$ computed in a point z of the polydisc Δ^n is

$$\mathcal{P}[f](z) \overset{\text{def}}{=} \int_{\mathbf{T}^n} f(x)P_z(x)d\nu(x)$$

where ν is the normalized Lebesgue measure on $\mathbf{T}^n$ and the *Poisson kernel* $P_z(x) \overset{\text{def}}{=} \prod_{i=1}^n \frac{1-|z_i|^2}{|x_i-z_i|^2}$ for all $x \in \mathbf{T}^n$. The main properties of the Poisson integral are (see for example [Rud2] p. 18):

(1) if f is continuous on $\overline{\Delta}^n$ and n-harmonic in Δ^n, that is, f is harmonic in each variable, then

$$\mathcal{P}[f](z) = f(z) \qquad \forall z \in \Delta^n; \tag{5}$$

(2) if $f \in L^1(\mathbf{T}^n, \mathbb{C})$ then

$$\lim_{r \to 1^-} \mathcal{P}[f](rw) = f(w) \text{ for almost all } w \in \mathbf{T}^n. \tag{6}$$

The following lemma will be useful later.

Lemma 5.3.7 [Ta2] *Let* $r, s \geq 1$ *be such that* $r + s = n$*, and let*

$$f(u,v) = \frac{M(u,v)\overline{Q}(\frac{1}{u}, \frac{1}{v})}{Q(u,v)} = \frac{P(u,v)}{Q(u,v)} : \overline{\Delta}^r \times \overline{\Delta}^s \to \overline{\Delta}$$

be a rational function of the form (4)*, where we identify* z *with* $(u,v) \in \overline{\Delta}^r \times \overline{\Delta}^s$*. If there exists* $(\tilde{u}, \tilde{v}) \in \mathbf{T}^r \times \Delta^s$ *such that* $|f(\tilde{u}, \tilde{v})| = 1$*, then* f *does not depend on the variable* v*.*

[1] $\frac{1}{z}$ stands for $(\frac{1}{z_1}, \ldots, \frac{1}{z_n})$.

Proof. Assuming $f(\tilde{u},\tilde{v}) = 1$, by the maximum principle $f(\tilde{u},\cdot) \equiv 1$. Write $Q(u,v)$ in the form $a(u)v^d + \ldots + b(u) \in \mathbb{C}[v_1,\ldots,v_s]$, where $v^d = v_1^{d_1}\ldots\ v_s^{d_s}$ with $d_j \in \mathbb{N}$, and $|d| \stackrel{\text{def}}{=} d_1 + \ldots\ + d_s$ is the degree of $Q(u,v)$ with respect to the variable v. By the hypothesis on Q, $Q(u,0) = b(u) \neq 0$ for all $u \in \mathbf{T}^r$.

Let $M(u,v) = e^{i\theta}u^{d''}v^{d'}$. Then $d'_i \geq d_i$ for $i = 1,\ldots,s$. If $|d'| > |d|$ then we should have $P(u,0) = 0$ for all $u \in \mathbf{T}^r$, whereas

$$P(\tilde{u},0) = f(\tilde{u},0)Q(\tilde{u},0) = b(\tilde{u}) \neq 0. \tag{7}$$

Hence, $d' = d$ and $P(u,0) = e^{i\theta}u^{d''}\overline{a}(\frac{1}{u}) = e^{i\theta}u^{d''}\overline{a(u)}$ for all $u \in \mathbf{T}^r$.

If $v(\zeta) \stackrel{\text{def}}{=} (\zeta,\ldots,\zeta) \in \overline{\Delta}^s$ for $\zeta \in \overline{\Delta}$, then

$$q(\zeta) \stackrel{\text{def}}{=} Q(\tilde{u},v(\zeta)) = a(\tilde{u})\zeta^{|d|} + \ldots + b(\tilde{u}) \in \mathbb{C}[\zeta].$$

If $|d| > 0$, the polynomial q has $|d|$ zeros in $\mathbb{C}$, and the absolute value of the product of such zeros is, by (7)

$$\left|\frac{b(\tilde{u})}{a(\tilde{u})}\right| = \left|\frac{P(\tilde{u},0)}{a(\tilde{u})}\right| = \left|\frac{e^{i\theta}\tilde{u}^{d''}\overline{a(\tilde{u})}}{a(\tilde{u})}\right| = 1.$$

Hence at least one zero, say $\tilde{\zeta}$ belongs to $\overline{\Delta}$, that is, $q(\tilde{\zeta}) = Q(\tilde{u},v(\tilde{\zeta})) = 0$, and we have a contradiction because $(\tilde{u},v(\tilde{\zeta})) \in \overline{\Delta}^n$. So $d = 0$, and f does not depend on the variable v. Q.E.D.

The following theorem generalizes (a) of Proposition 5.2.3.

Theorem 5.3.8 [Ta2] *If $F \in \mathcal{I}(\Delta^n)$ and $F_{|\mathbf{T}^n} \in \mathcal{E}^1(\mathbf{T}^n)$ then F has one and only one fixed point in Δ^n.*

Proof. Assume that F has no fixed point in Δ^n. The sequence $\{F^k\}$ of the iterates of F is divergent on the compact sets of Δ^n and there exists a subsequence $\{F^{k_j}\}$ converging to a map in $\mathrm{Hol}(\Delta^n, \partial(\overline{\Delta}^n))$ (see [Ab] p. 274).

By the maximum principle, at least one component of this limit map, say the first, is identically equal to $c \in \mathbf{T}^1$. Hence the subsequence of the first components $\{(F^{k_j})_1\}$ converges uniformly on the compact sets of Δ^n to the constant c.

Put $p_m(\zeta) \stackrel{\text{def}}{=} \zeta^m$ with $m \in \mathbb{N}$ and $\zeta \in \overline{\Delta}$; then $p_m \circ (F^{k_j})_1$ is holomorphic in Δ^n and continuous on $\overline{\Delta}^n$. Therefore we can apply (5) and, for all $z \in \Delta^n$, we have

$$\int_{\mathbf{T}^n} p_m((F^{k_j})_1(x))P_z(x)d\nu(x) = \mathcal{P}[p_m \circ (F^{k_j})_1](z) = p_m((F^{k_j})_1(z)). \tag{8}$$

Now, the limit of (8), for $j \to \infty$ equals $p_m(c)$. Since the complex vector space generated by the functions p_m, $\overline{p_m}$ for all $m \in \mathbb{N}$ is dense in $C(\mathbf{T}^1, \mathbb{C})$ (it is the space of all trigonometric polynomials, see [Rud1] p. 88), for all $g \in C(\mathbf{T}^1, \mathbb{C})$

$$\lim_{j\to\infty} \int_{\mathbf{T}^n} g((F^{k_j})_1(x))P_z(x)d\nu(x) = g(c). \tag{9}$$

Note that the complex vector space generated by the Poisson kernels P_z for all $z \in \Delta^n$ is dense in $L^1(\mathbf{T}^n, \mathbb{C})$, because, if $f \in L^\infty(\mathbf{T}^n, \mathbb{C})$, is such that

$$\int_{\mathbf{T}^n} f(x)P_z(x)d\nu(x) = 0 \qquad \forall z \in \Delta^n,$$

then, by (6), $f(x) = 0$ for almost every $x \in \mathbf{T}^n$. Density follows from the fact that $L^\infty(\mathbf{T}^n, \mathbb{C})$ is the dual space of $L^1(\mathbf{T}^n, \mathbb{C})$.

Therefore, by (9) and the identity $\int_{\mathbf{T}^n} P_z(x)d\nu(x) = 1$, the following equation holds for all $h \in L^1(\mathbf{T}^n, \mathbb{C})$ and $g \in C(\mathbf{T}^1, \mathbb{C})$:

$$\lim_{j\to\infty} \int_{\mathbf{T}^n} g((F^{k_j})_1(x))h(x)d\nu(x) = g(c) \int_{\mathbf{T}^n} h(x)d\nu(x). \tag{10}$$

Since $F_{|\mathbf{T}^n}$ is C^∞ and differentiably expanding, by Theorem 2.2.4 there exists a normalized invariant measures μ_F equivalent to the Lebesgue measure ν. Therefore, by the Radon-Nikodym theorem, it is possible to find $h \in L^1(\mathbf{T}^n, \mathbb{R})$ such that $d\mu_F = hd\nu$; thus, by (10),

$$\lim_{j\to\infty} \int_{\mathbf{T}^n} g((F^{k_j})_1(x))d\mu_F(x) = g(c) \int_{\mathbf{T}^n} h(x)d\nu(x) = g(c)\mu_F(\mathbf{T}^n) = g(c). \tag{11}$$

By the invariance of the measure μ_F, for all $j \in \mathbb{N}$

$$\int_{\mathbf{T}^n} g((F^{k_j})_1(x))d\mu_F(x) = \int_{\mathbf{T}^n} g(x_1)d\mu_F(x) \quad \forall g \in C(\mathbf{T}^1, \mathbb{C}),$$

and, by (11), this implies

$$\int_{\mathbf{T}^n} g(x_1)d\mu_F(x) = g(c) \qquad \forall g \in C(\mathbf{T}^1, \mathbb{C}). \tag{12}$$

Let $\{g_i\}$ be a bounded sequence in $C(\mathbf{T}^1, \mathbb{C})$ that converges pointwise to I_c, the characteristic function of the set $\{c\}$, and let $E \stackrel{\text{def}}{=} \{c\} \times \mathbf{T}^{n-1}$. Then

$$\lim_{i\to\infty} \int_{\mathbf{T}^n} g_i(x_1)d\mu_F(x) = \lim_{i\to\infty} g_i(c) = I_c(c) = 1. \tag{13}$$

On the other hand, since $\nu(E) = 0$ also $\mu_F(E) = 0$ (because μ_F and ν are equivalent), and (12) yields

$$\lim_{i\to\infty} \int_{\mathbf{T}^n} g_i(x_1)d\mu_F(x) = \int_{\mathbf{T}^n} I_c(x_1)d\mu_F(x) = \int_{\mathbf{T}^n} I_E(x)d\mu_F(x) = \mu_F(E) = 0,$$

which contradicts (13). Hence F must have at least one fixed point in Δ^n.

An inductive argument will show now that the existence of at least two fixed points in Δ^n leads to a contradiction. If $n = 1$, then, by Proposition 5.2.3, F has a unique fixed point in Δ.

Now suppose that uniqueness has been established for $1 \leq r < n$. Since Δ^n is homogeneous, we can assume that one of the fixed points is in 0. Let $a \in \Delta^n \setminus \{0\}$ be another fixed point.

By (b) of Theorem 5.3.2, there exists a complex geodesic $\varphi \in \mathrm{Hol}(\Delta, \Delta^n)$ such that

$$0, a \in \varphi(\Delta) \subset \mathrm{Fix}_{\Delta^n}(F)$$

At least one component of φ, say the first, is an automorphism of Δ, hence, the closure of the range of this complex geodesic intersects the boundary of the polydisc in at least one point $\tilde{z} \in \mathbf{T}^1 \times \overline{\Delta}^{n-1}$. Therefore, by permuting the last $n-1$ variables, we can assume that $\tilde{z}$ is of the form $(\tilde{u}, \tilde{v}) \in \mathbf{T}^r \times \Delta^s$ with $r \geq 1$, $s \geq 0$ and $r + s = n$.

Since F is continuous on $\overline{\Delta}^n$ we have that $F(\tilde{u},\tilde{v}) = (\tilde{u},\tilde{v})$. At each fixed point of F in Δ^n, the eigenvalues of the differential of F have absolute value less or equal to 1 (see [Ve] p. 92), hence

$$|\det(D_{\varphi(\zeta)}F)| \leq 1 \;\; \forall \zeta \in \Delta \;\text{ implies that }\; |\det(D_{(\tilde{u},\tilde{v})}(F))| \leq 1.$$

On the other hand, since $F_{|\mathbf{T}^n} \in \mathcal{E}^1(\mathbf{T}^n)$, $|\det(D_{(u,v)}F)| > 1$ for each fixed point $(u,v) \in \mathbf{T}^n$; therefore the intersection $(\tilde{u},\tilde{v})$ can not belong to the Shilov boundary $\mathbf{T}^n$. Then, since $f_i(\tilde{u},\tilde{v}) = \tilde{u}_i$ for $i = 1,\ldots,r$, by Lemma 5.3.7 the first r components $f_1,\ldots,f_r$ of F, do not depend on the last s variables. So we can define the map $F_1 = (f_1,\ldots,f_r) \in \mathcal{I}(\Delta^r)$ that has 0 and $(\varphi_1(\frac{1}{2}),\ldots,\varphi_r(\frac{1}{2})) \in \Delta^r$ as fixed points. Note that these two fixed points are different because φ_1 is an automorphism of Δ which fixes 0. Therefore $\varphi_1(\frac{1}{2}) \neq 0$. Moreover,

$$D_{(u,v)}F = \begin{vmatrix} D_uF_1 & 0 \\ * & * \end{vmatrix} \qquad \forall (u,v) \in \mathbf{T}^r \times \mathbf{T}^s.$$

Since $F_{|\mathbf{T}^n} \in \mathcal{E}^1(\mathbf{T}^n)$, the matrix $D_{(u,v)}F$ is invertible. Thus, if $w \in T_u(\mathbf{T}^r)$, there exists $w' \in \mathbb{R}^s$ such that

$$D_{(u,v)}F(w,w') = (D_uF_1w, 0).$$

Therefore, F_1 is differentiably expanding on $\mathbf{T}^r$ because

$$\|D_uF_1w\| = \|D_{(u,v)}F(w,w')\| \geq \lambda\|(w,w')\| \geq \lambda\|w\|,$$

where $\lambda > 1$ is the expanding constant of F, and this contradicts the inductive assumption. Q.E.D.

Remark 5.3.9 However, for $n > 1$, unlike the case $n = 1$, it is not true that if $F \in \mathcal{I}(\Delta^n)$ is non-invertible and has one and only one fixed point in Δ^n then $F_{|\mathbf{T}^n} \in \mathcal{E}^1(\mathbf{T}^n)$. In fact, for $n = 2$ it is easy to verify that the map

$$F(z_1,z_2) = (z_1^2z_2, z_1z_2) \qquad \forall (z_1,z_2) \in \overline{\Delta}^2,$$

belongs to $\mathcal{I}(\Delta^n)$, is non-invertible and its unique fixed point in Δ^n is $(0,0)$. But $F_{|\mathbf{T}^2}$ is not differentiably expanding because

$$F(1,1) = (1,1) \in \mathbf{T}^2, \quad D_{(1,1)}F = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

and $\det(D_{(1,1)}F) = 1$.

Definition 5.3.10 A *boundary component* C of the polydisc Δ^n is defined as the cartesian product, in some order, of $r \geq 1$ copies of $\mathbf{T}^1$ and $s \geq 0$ copies of Δ with $r + s = n$. The boundary of the polydisc is partitioned in $2^n - 1$ boundary components.

If $F \in \mathcal{I}(\Delta^n)$ and $F_{|\mathbf{T}^n} \in \mathcal{E}^1(\mathbf{T}^n)$, we already know that F has $\mathcal{N}(F_{|\mathbf{T}^n}) \geq 1$ fixed points on $\mathbf{T}^n$ and a unique fixed point in Δ^n. The following theorem gives an estimate of the number of fixed points of F on the other boundary components.

Moreover, also the case of two commuting maps will be investigated. Adding the differentiably expanding hypothesis, our result turns out to be more precise, about the number of common fixed points and their location in $\overline{\Delta}^n$, than the theorem proved by L. F. Heath and T. J. Suffridge in [HeSu], which assures that there exists at least a common fixed point for two commuting maps $F, G \in \mathrm{Hol}(\Delta^n, \Delta^n) \cap C(\overline{\Delta}^n, \overline{\Delta}^n)$.

Theorem 5.3.11 *For $F \in \mathcal{I}(\Delta^n)$ such that $F_{|\mathbf{T}^n} \in \mathcal{E}^1(\mathbf{T}^n)$, the following facts holds:*

(a) if C is a boundary component such that $F(C) \cap C \neq \emptyset$ then

$$F(C) \subset C \quad \textit{and} \quad 1 \leq \mathrm{card}(\mathrm{Fix}_C(F)) \leq \mathcal{N}(F_{|\mathbf{T}^n}),$$

and therefore $1 + \mathcal{N}(F_{|\mathbf{T}^n}) \leq \mathrm{card}(\mathrm{Fix}_{\overline{\Delta}^n}(F)) \leq 1 + (2^n - 1)\mathcal{N}(F_{|\mathbf{T}^n})$;

(b) if $G \in \mathcal{I}(\Delta^n)$, $G_{|\mathbf{T}^n} \in \mathcal{E}^1(\mathbf{T}^n)$ commutes with $F_{|\mathbf{T}^n}$, and, the cardinalities of the fixed points on $\mathbf{T}^n$ of F and G are relatively prime, then F

and G *have a unique fixed point in every boundary component* C *such that* $F(C) \subset C$ *and* $G(C) \subset C$.

Therefore $2 \leq \operatorname{card}(\operatorname{Fix}_{\overline{\Delta}^n}(F)) \cap \operatorname{Fix}_{\overline{\Delta}^n}(G)) \leq 2^n$.

Proof. (a) When $C = \mathbf{T}^n$ we already know that $\operatorname{card}(\operatorname{Fix}_C(F)) = \mathcal{N}(F_{|\mathbf{T}^n})$. Thus we can assume that $C = \mathbf{T}^r \times \Delta^s$ with $r \geq 1$, $s \geq 1$ and $r + s = n$. If $(\tilde{u}, \tilde{v}) \in F(C) \cap C$ then, since $|f_i(\tilde{u}, \tilde{v})| = 1$ for $i = 1, \ldots, r$, by Lemma 5.3.7 the first r component maps $f_1, \ldots, f_r$ of F, do not depend on the last s variables. Therefore we can define the map $F_1 = (f_1, \ldots, f_r) \in \mathcal{I}(\Delta^r)$.

As in the preceding theorem, F_1 is differentiably expanding on $\mathbf{T}^r$ and, if Φ_A is the endomorphism associated to F, then $A = \begin{vmatrix} A_1 & 0 \\ * & * \end{vmatrix}$ and we associate Φ_{A_1} to F_1. This means that the cardinality of the fixed points of F_1 on $\mathbf{T}^r$ is $|\det(A_1 - \mathbb{I})| \geq 1$ which divides $\mathcal{N}(F_{|\mathbf{T}^n}) = |\det(A - \mathbb{I})|$.

Now, let $\tilde{u} \in \mathbf{T}^r$ be one of these fixed points and define

$$F_2(\cdot) = (f_{r+1}(\tilde{u}, \cdot), \ldots, f_n(\tilde{u}, \cdot)) \in \mathcal{I}(\Delta^s).$$

Then F_2 is differentiably expanding on $\mathbf{T}^s$. In fact

$$D_{(\tilde{u},v)}F = \begin{vmatrix} D_{\tilde{u}}F_1 & 0 \\ * & D_v F_2 \end{vmatrix} \qquad \forall v \in \mathbf{T}^s$$

and since $F_{|\mathbf{T}^n} \in \mathcal{E}^1(\mathbf{T}^n)$, if $w' \in T_v(\mathbf{T}^s)$, then

$$\|D_v F_2 w'\| = \|D_{(\tilde{u},v)}F(0, w')\| \geq \lambda \|(0, w')\| \geq \lambda \|w'\|.$$

Thus by the preceding theorem F_2 has a unique fixed point $\tilde{v} \in \Delta^s$, that is

$$F(\tilde{u}, \tilde{v}) = (F_1(\tilde{u}), F_2(\tilde{v})) = (\tilde{u}, \tilde{v}) \in C.$$

Hence, $1 \leq \operatorname{card}(\operatorname{Fix}_C(F)) = |\det(A_1 - \mathbb{I})| \leq \mathcal{N}(F_{|\mathbf{T}^n})$.

(b) Assume that $C = \mathbf{T}^r \times \Delta^s$ with $r \geq 1$, $s \geq 0$ and $r + s = n$. Since C is invariant for both F and G, as in (a), we can define the maps F_1 and G_1 belonging to $\mathcal{I}(\Delta^r)$, differentiably expanding on $\mathbf{T}^r$. Moreover, F_1 and G_1

commute, and since the cardinalities of the fixed points of F_1 and G_1 on $\mathbf{T}^r$ divide respectively $\mathcal{N}(F_{|\mathbf{T}^n})$ and $\mathcal{N}(G_{|\mathbf{T}^n})$, they are relatively prime. Thus, by Theorem 5.3.8, F_1 and G_1 have a unique common fixed point $\tilde{u} \in \mathbf{T}^r$.

Now, as in (a) we define the maps F_2 and G_2 belonging to $\mathcal{I}(\Delta^s)$, differentiably expanding on $\mathbf{T}^s$. By the preceding theorem, F_2 has a unique fixed point $\tilde{v} \in \Delta^s$ and since F_2 and G_2 commute, then $G_2(\tilde{v}) = \tilde{v}$. Therefore $(\tilde{u}, \tilde{v})$ is the unique common fixed point of F and G in C. Q.E.D.

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