

ANALISI MATEMATICA 2 - LEZIONE 16

ALCUNI ESERCIZI DEL FOGLIO 4

1.a $2y - x^2 = 2 + \log(y)$ in $(0,1)$. T_2 di $y = \varphi(x)$ in $x=0$?

Sia $f(x,y) = 2y - x^2 - 2 - \log(y)$.

$$f_y(0,1) = \left(2 - 2x - \frac{1}{y}\right)\bigg|_{(0,1)} = 1 \neq 0 \stackrel{\text{TFI}}{\Rightarrow} \exists y = \varphi(x).$$

$$\varphi(0) = 1, \quad \varphi'(0) = -\frac{f_x(0,1)}{f_y(0,1)} = -\frac{(-2x)}{1}\bigg|_{(0,1)} = 0.$$

Inoltre

$$2\varphi(x) - x^2 = 2 + \log(\varphi(x)) \stackrel{D}{\Rightarrow} 2\varphi'(x) - 2x = \frac{\varphi'(x)}{\varphi(x)}$$

$$\stackrel{D}{\Rightarrow} 2\varphi''(x) - 2 = \frac{\varphi''(x)\varphi(x) - (\varphi'(x))^2}{(\varphi(x))^2}$$

Per $x=0$ otteniamo

$$2\varphi''(0) - 2 = \frac{\varphi''(0) \cdot 1 - 0^2}{1^2} \Rightarrow \varphi''(0) = 2.$$

Quindi

$$T_2(x) = 1 + 0 \cdot x + \frac{2}{2}x^2 = 1 + x^2.$$

2.a $\begin{cases} xyz = -2 \\ x^3 + y^3 + z^3 = 8 \end{cases}$ in $(-1, 2, 1)$. T_1 di $y = \varphi(x)$ e $z = \psi(x)$ in $x = -1$?

Siamo $f(x,y,z) = xyz + 2 = 0$, $g(x,y,z) = x^3 + y^3 + z^3 - 8 = 0$.

$$J = \begin{bmatrix} f_y & f_z \\ g_y & g_z \end{bmatrix} = \begin{bmatrix} xz & xy \\ 3y^2 & 3z^2 \end{bmatrix} \stackrel{(-1,2,1)}{\Rightarrow} \begin{bmatrix} -1 & -2 \\ 12 & 3 \end{bmatrix} \Rightarrow \det(J) = 21 \neq 0$$

Quindi per TFI $\exists y = \varphi(x)$ e $\exists z = \psi(x)$: $\varphi(-1) = 2$, $\psi(-1) = 1$.

Inoltre derivando

$$\begin{cases} x\varphi(x)\psi(x) = -2 \\ x^3 + (\varphi(x))^3 + (\psi(x))^3 = 8 \end{cases}$$

si ottiene

$$\begin{cases} \varphi(x)\psi(x) + x\varphi'(x)\psi(x) + x\varphi(x)\psi'(x) = 0 \\ 3x^2 + 3(\varphi(x))^2 \cdot \varphi'(x) + 3(\psi(x))^2 \cdot \psi'(x) = 0 \end{cases}$$

Per $x = -1$ si ha

$$\begin{cases} 2 \cdot 1 - \varphi'(-1) \cdot 1 - 2\psi'(-1) = 0 \\ 3 + 12\varphi'(-1) + 3\psi'(-1) = 0 \end{cases} \quad \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \varphi'(-1) \\ \psi'(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

da cui

$$\varphi'(-1) = \frac{\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}} = \frac{4}{-7} \quad \text{e} \quad \psi'(-1) = \frac{\begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}} = \frac{-9}{-7} = \frac{9}{7}.$$

Infine

$$\varphi) T_1(x) = 2 - \frac{4}{7}(x+1) \quad \psi) T_1(x) = 1 + \frac{9}{7}(x+1).$$

3.2

$$\{(x, y, z) : xyz = -8\}$$

Piano tangente
in $(-2, 1, 4)$?

Sia $f(x, y, z) = xyz + 8$. Allora

$$\nabla f(x, y, z) = (yz, xz, xy)$$

e il piano tangente in $(-2, 1, 4)$ è

$$f_x(-2, 1, 4)(x+2) + f_y(-2, 1, 4)(y-1) + f_z(-2, 1, 4)(z-4) = 0$$

$$\text{ossia } 4(x+2) - 8(y-1) - 2(z-4) = 0$$

$$2x - 4y - z + 12 = 0.$$

4.a $f(x,y) = xy + x^2$ Punti stazionari vincolati
in $\Gamma = \{(x,y) : x^2 + y^2 + xy = 4\}$?

Sia $g(x,y) = x^2 + y^2 + xy - 4 = 0$ allora

$$\nabla g(x,y) = (2x+y, 2y+x) = (0,0) \iff (x,y) = (0,0) \notin \Gamma$$

Tutti i punti di Γ sono regolari

Punti stazionari vincolati

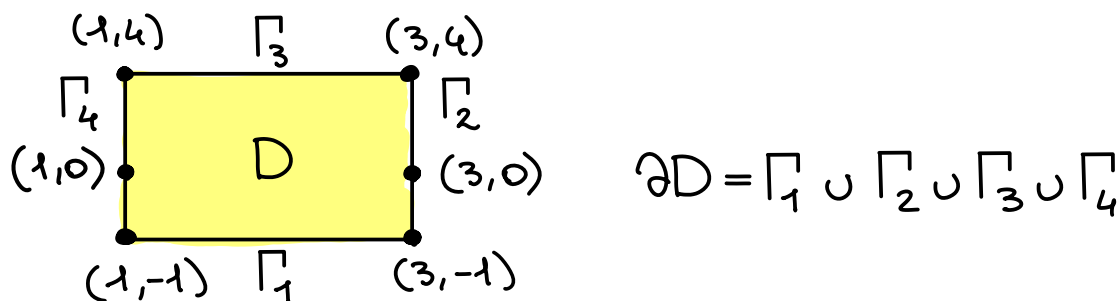
$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases} \begin{cases} y + 2x = \lambda(2x + y) \\ x = \lambda(2y + x) \\ x^2 + y^2 + xy = 4 \end{cases} \rightarrow \begin{matrix} (\lambda-1)(2x+y) = 0 \\ \lambda = 1 \quad y = -2x \end{matrix}$$

$$\begin{cases} \lambda = 1 \\ x = 2y + x \rightarrow y = 0 \\ x^2 + y^2 + xy = 4 \rightarrow x^2 = 4 \rightarrow x = \pm 2 \end{cases} \Rightarrow \begin{matrix} (2,0) \\ (-2,0) \end{matrix} \quad \lambda = 1$$

$$\begin{cases} y = -2x \\ x = \lambda(-4x + x) \rightarrow \lambda = -\frac{1}{3} \\ x^2 + 4x^2 - 2x^2 = 4 \rightarrow 3x^2 = 4 \rightarrow x = \pm \frac{2}{\sqrt{3}} \end{cases} \Rightarrow \begin{matrix} (\frac{2}{\sqrt{3}}, -\frac{4}{\sqrt{3}}) \\ (-\frac{2}{\sqrt{3}}, \frac{4}{\sqrt{3}}) \end{matrix} \quad \lambda = -\frac{1}{3}$$

5.a $f(x,y) = \log((x+1)^2 + y^2)$ Max/min in
 $D = [1,3] \times [-1,4]$?

$f \in C^2(\mathbb{R}^2 \setminus \{(-1,0)\})$, D è compatto e $(-1,0) \notin D$.



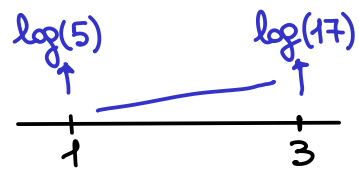
Punti stazionari interni a D :

$$\nabla f(x,y) = \left(\frac{2(x+1)}{(x+1)^2 + y^2}, \frac{2y}{(x+1)^2 + y^2} \right) = (0,0) \quad \text{Senza soluzioni in } \mathbb{R}^2 \setminus \{(-1,0)\}$$

Bordo di D : $\partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$

1) $\Gamma_1 = \{(x, -1) : x \in [1, 3]\}$,

$$h(x) = f(x, -1) = \log((x+1)^2 + 1)$$

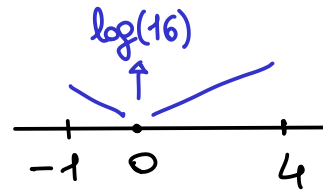


$$f(1, -1) = \log(5), \quad f(3, -1) = \log(17)$$

2) $\Gamma_2 = \{(3, y) : y \in (-1, 4)\}$,

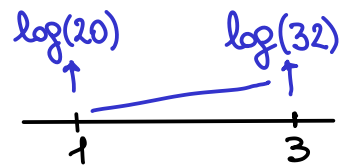
$$h(y) = f(3, y) = \log(16 + y^2)$$

$$f(3, 0) = \log(16)$$



3) $\Gamma_3 = \{(x, 4) : x \in [1, 3]\}$,

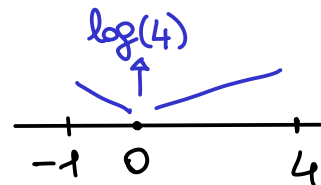
$$h(x) = f(x, 4) = \log((x+1)^2 + 16)$$



$$f(1, 4) = \log(20), \quad f(3, 4) = \log(32)$$

4) $\Gamma_4 = \{(1, y) : y \in (-1, 4)\}$,

$$h(y) = f(1, y) = \log(4 + y^2)$$



$$f(1, 0) = \log(4)$$

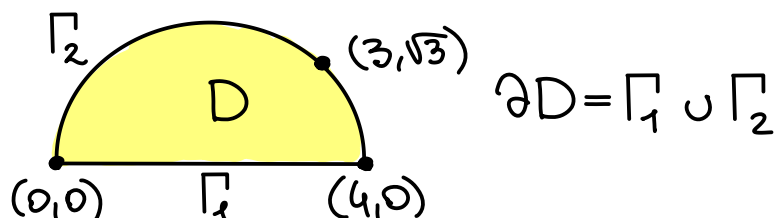
Confrontando i valori trovati si ha che $(1, 0)$ è un punto di minimo assoluto con valore $\log(4)$ e $(3, 4)$ è un punto di massimo assoluto con valore $\log(32)$.

5.d

$$f(x,y) = x + \sqrt{3}y$$

Max/min in
 $D = \{(x,y) : x^2 + y^2 \leq 4x, y \geq 0\} ?$
 $\hookrightarrow (x-2)^2 + y^2 \leq 4$

$f \in C^2(\mathbb{R}^2)$ e D è compatto



Non ci sono punti stazionari interni perché

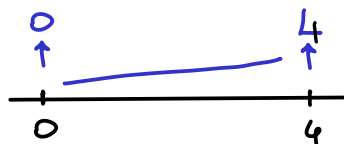
$$\nabla f(x,y) = (1, \sqrt{3}) \neq (0,0)$$

Bordo di D : $\partial D = \Gamma_1 \cup \Gamma_2$

$$f(0,0) = 0, f(4,0) = 4$$

1) $\Gamma_1 = \{(x,0) : x \in [0,4]\}$

$$h(x) = f(x,0) = x$$



2) $\Gamma_2 = \{(x,y) : x^2 + y^2 - 4x = 0, y > 0\}$

$$\nabla g(x,y) = (2x-4, 2y) = (0,0) \Rightarrow (x,y) = (2,0) \notin \Gamma_2 \quad \text{tutti regolari}$$

Punti stazionari vincolati: $\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$

$$\begin{cases} 1 = \lambda(2x-4) \rightarrow y = \sqrt{3}(x-2) \\ \sqrt{3} = \lambda(2y) \rightarrow \lambda = \frac{\sqrt{3}}{2y} \\ x^2 + y^2 = 4x \rightarrow x^2 + 3(x-2)^2 = 4x, x^2 + 3x^2 - 12x + 12 = 4x \\ x^2 - 4x + 3 = 0 \rightarrow x = 2 \pm 1 = \frac{1}{3} \end{cases}$$

$$\begin{cases} x=1 \\ y=-\sqrt{3} \\ \lambda=-\frac{1}{2} \end{cases} \cup \begin{cases} x=3 \\ y=\sqrt{3} \\ \lambda=\frac{1}{2} \end{cases} \quad \begin{matrix} (1, -\sqrt{3}) \notin \Gamma_2 \\ (3, \sqrt{3}) \in \Gamma_2 \end{matrix}$$

$$f(3, \sqrt{3}) = 6$$

Confrontando i valori trovati si ha che $(0,0)$ è un punto di minimo assoluto con valore 0 e $(3, \sqrt{3})$ è un punto di massimo assoluto con valore 6.