

ANALISI MATEMATICA 2 - FOGLIO 2

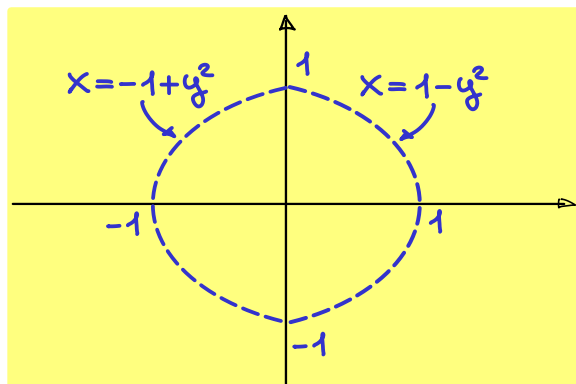
1.a

$$f(x,y) = \frac{e^{xy}}{|x|+y^2-1} \quad \text{Dominio } D?$$

$$D = \{(x,y) \in \mathbb{R}^2 : |x|+y^2-1 \neq 0\}$$

Notiamo che

$$|x|+y^2-1=0 \iff |x|=1-y^2 \geq 0 \iff \begin{cases} x = \pm(1-y^2) \\ \text{con } |y| \leq 1 \end{cases}$$



parabole

↑ striscie orizzontali

1.b

$$f(x,y) = \frac{1}{\sqrt{2-|x|-|y|}} \quad \text{Dominio } D?$$

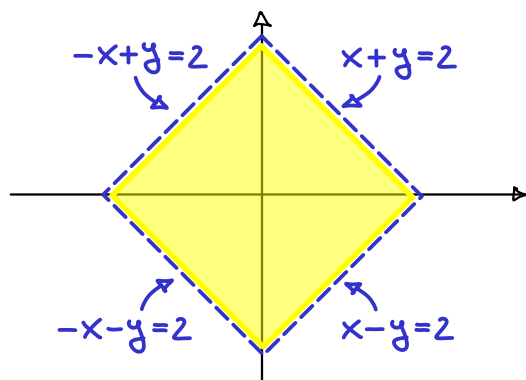
$$D = \{(x,y) \in \mathbb{R}^2 : |x|+|y| < 2\}$$

Se $x \geq 0$ e $y \geq 0$ $|x|+|y|=2 \iff x+y=2$ retta

Se $x \geq 0$ e $y \leq 0$ $|x|+|y|=2 \iff x-y=2$ retta

Se $x \leq 0$ e $y \geq 0$ $|x|+|y|=2 \iff -x+y=2$ retta

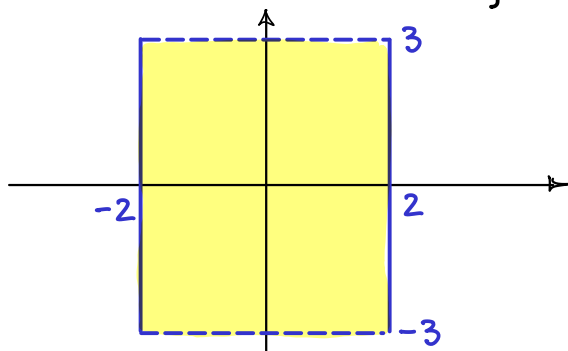
Se $x \leq 0$ e $y \leq 0$ $|x|+|y|=2 \iff -x-y=2$ retta



1.c

$$f(x,y) = \frac{\sqrt{4-x^2}}{\sqrt{9-y^2}} \quad \text{Dominio } D?$$

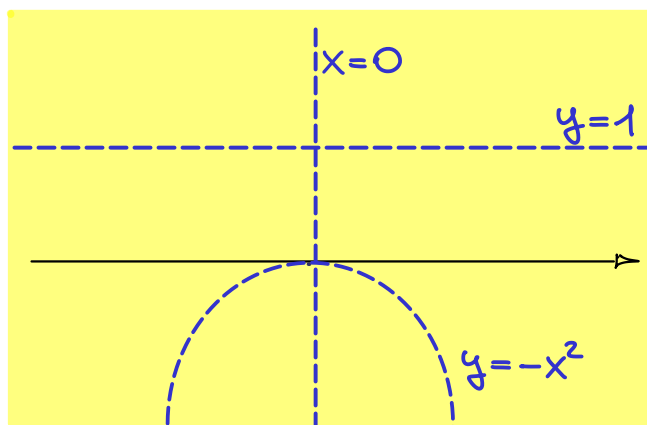
$$D = \{(x,y) \in \mathbb{R}^2 : |x| \leq 2 \text{ e } |y| < 3\} = [-2, 2] \times (-3, 3)$$



1.d

$$f(x,y) = \frac{\log|x^3 + x(x^2+y)|}{y-1} \quad \text{Dominio } D?$$

$$D = \{(x,y) \in \mathbb{R}^2 : x \neq 0 \text{ e } x^2 + y \neq 0 \text{ e } y \neq 1\}$$



1.e

$$f(x,y) = \log(\sin(\pi\sqrt{x^2+y^2})) \quad \text{Dominio } D?$$

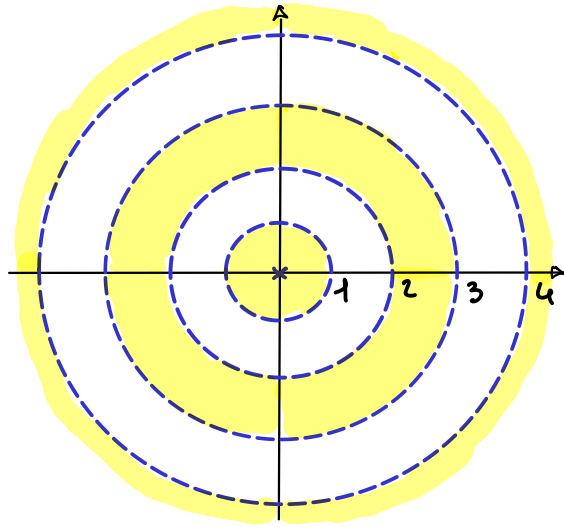
$$D = \{(x,y) : \sin(\pi\sqrt{x^2+y^2}) > 0\}$$

$$= \bigcup_{k=0}^{\infty} \{(x,y) : 2k < \sqrt{x^2+y^2} < 2k+1\}$$

← quello centrato in (0,0) di oggi
2k e 2k+1 senza bordo

$$\sin(t) > 0 \text{ per } t \in \bigcup_{k \in \mathbb{Z}} (2k\pi, (2k+1)\pi)$$

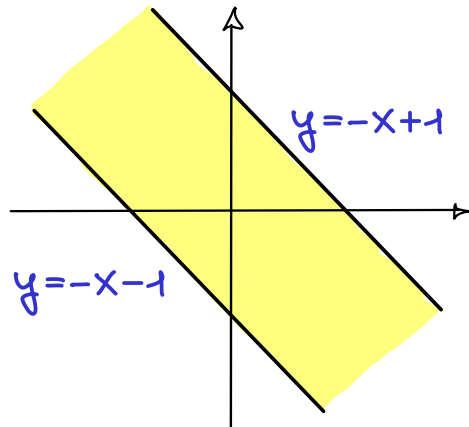
-2π + -π 0 + π 2π 3π



1.ª

$f(x,y) = \arcsen(x+y)$ Dominio D?

$$D = \{(x,y) : -1 \leq x+y \leq 1\}$$

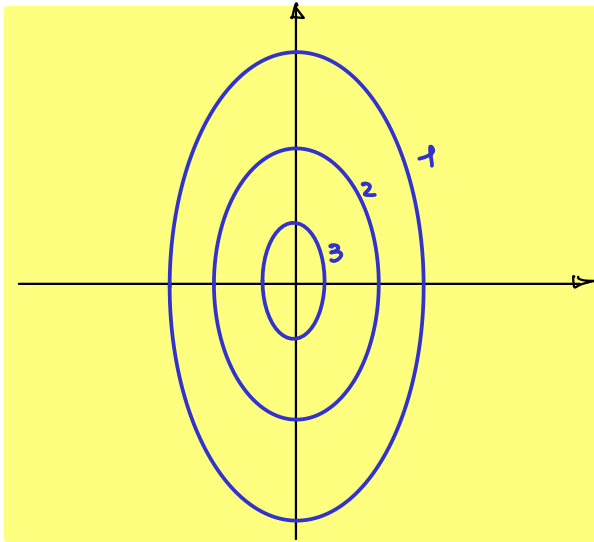


2.a

$$f(x,y) = \frac{1}{\sqrt{4x^2+y^2}} \quad \text{Curve di livello?}$$

$$\text{Per } (x,y) \in D = \mathbb{R}^2 \setminus \{(0,0)\},$$

$$\frac{1}{\sqrt{4x^2+y^2}} = K > 0 \iff \frac{1}{K^2} = 4x^2 + y^2 \iff \frac{x^2}{\left(\frac{1}{2K}\right)^2} + \frac{y^2}{\left(\frac{1}{K}\right)^2} = 1$$



ellisse centrata in (0,0)
con semiasse $\frac{1}{2K}$ e $\frac{1}{K}$

Al crescere di K
le ellissi diventano
sempre più piccole

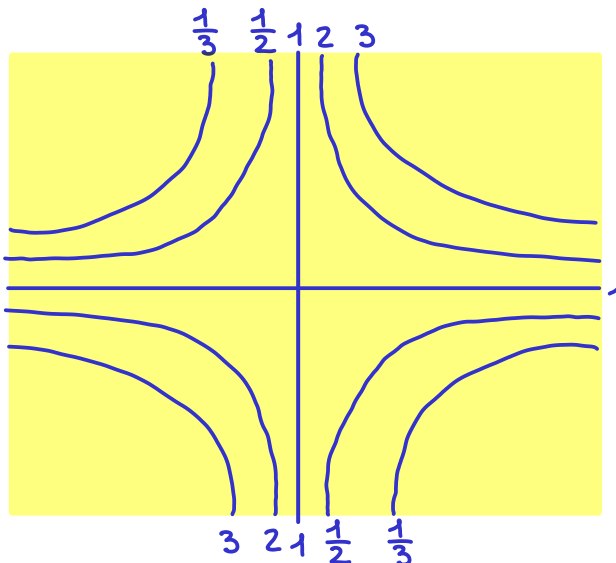
2.b

$$f(x,y) = e^{xy} \quad \text{Curve di livello?}$$

$$\text{Per } (x,y) \in D = \mathbb{R}^2$$

$$e^{xy} = K > 0 \iff y = \log(K) \Rightarrow \begin{cases} x=0 \text{ oppure } y=0 \text{ se } K=1 \\ y = \frac{\log(K)}{x} \text{ altrimenti} \end{cases}$$

iperbole



Per $K=1$ c'è la coppia
di rette $x=0$ e $y=0$
Le iperboli stanno
nel 1° e 3° quadrante
per $K>1$ e nel 2° e 4°
quadrante per $0<K<1$.

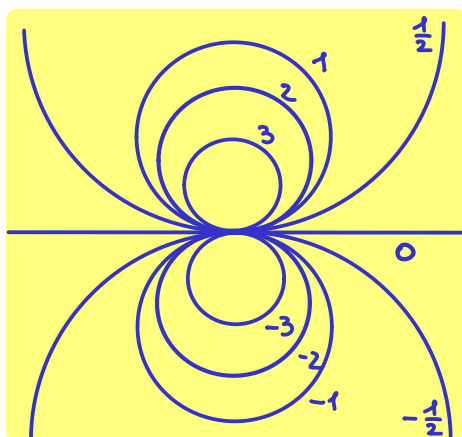
2.c

$$f(x,y) = \frac{4y}{x^2+y^2} \quad \text{Curve di livello?}$$

Per $(x,y) \in D = \mathbb{R}^2 \setminus \{(0,0)\}$,

$$\frac{4y}{x^2+y^2} = k \quad \begin{matrix} k \neq 0 \\ \Leftrightarrow x^2+y^2 - \frac{4y}{k} = 0 \Leftrightarrow x^2 + (y - \frac{2}{k})^2 = \frac{4}{k^2} \\ k=0 \Rightarrow y=0 \end{matrix}$$

circonferenza di centro $(0, \frac{2}{k})$ e raggio $\frac{2}{k}$



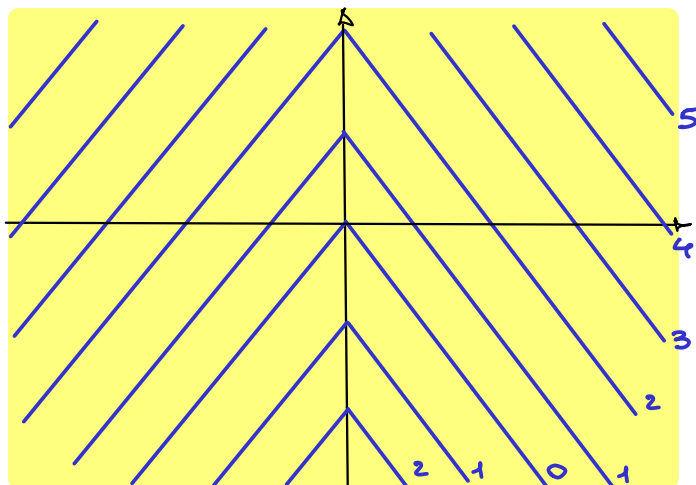
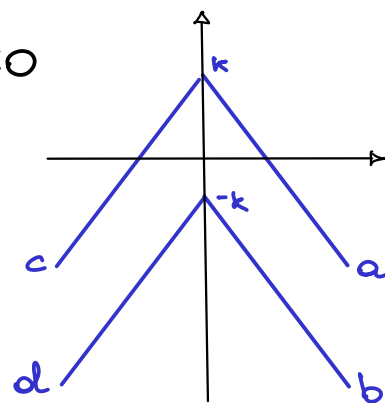
2.d

$$f(x,y) = |x| + y \quad \text{Curve di livello?}$$

Per $(x,y) \in D = \mathbb{R}^2$,

$$|x| + y = k \geq 0 \Leftrightarrow \begin{cases} |x+y| = k & \text{se } x \geq 0 \\ |-x+y| = k & \text{se } x \leq 0 \end{cases}$$

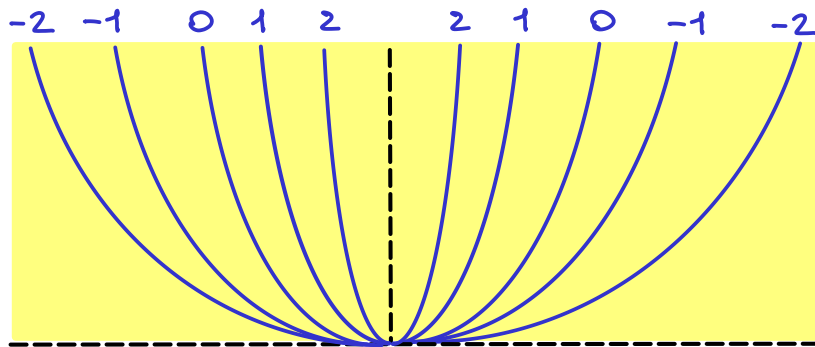
$$\Leftrightarrow \begin{cases} x+y=k & \text{se } x \geq 0 & \text{a)} \\ x+y=-k & \text{se } x \geq 0 & \text{b)} \\ -x+y=k & \text{se } x \leq 0 & \text{c)} \\ -x+y=-k & \text{se } x \leq 0 & \text{d)} \end{cases}$$



2.e $f(x,y) = \log\left(\frac{y}{x^2}\right)$ Curve di livello?

Per $(x,y) \in D = \{(x,y) : x \neq 0, y > 0\}$

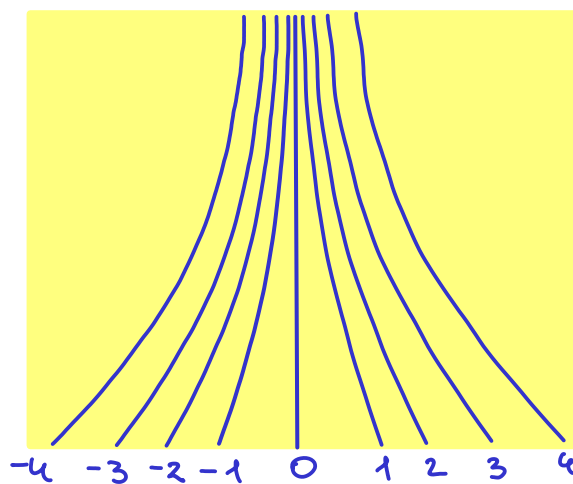
$$\log\left(\frac{y}{x^2}\right) = k \iff y = \underbrace{e^k}_{>0} x^2$$



2.f $f(x,y) = (x+1)e^y$ Curve di livello?

Per $(x,y) \in D = \mathbb{R}^2$,

$$(x+1)e^y = k \iff x = -1 + k e^{-y}$$



3.a $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{|x|+|y|} = 0$

Nel dominio $D = \mathbb{R}^2 \setminus \{(0,0)\}$ se restringiamo f lungo le rette $x=0$ e $y=mx$ otteniamo sempre 0:

$$f(0,y) = \frac{0}{|y|} = 0 \rightarrow 0 \text{ e } f(x, mx) = \frac{mx^2}{|x|(1+|m|)} \rightarrow 0.$$

Quindi se il limite esiste vale 0.

Passando alle coordinate polari $x = \rho \cos \theta$ e $y = \rho \sin \theta$

$$|f(x,y)| = \frac{\rho^2 |\cos \theta \sin \theta|}{\rho (|\cos \theta| + |\sin \theta|)} \leq \rho \cdot \frac{1}{m} \rightarrow 0 \text{ per } \rho \rightarrow 0$$

dove

$$m = \min_{\theta \in [0, 2\pi]} (|\cos \theta| + |\sin \theta|) > 0 \quad \text{perché la funzione } \theta \rightarrow |\cos \theta| + |\sin \theta| > 0 \text{ è continua nel compatto } [0, 2\pi]$$

In alternativa si può anche dire che per $(x,y) \rightarrow (0,0)$

$$0 \leq \frac{|xy|}{|x|+|y|} = |x| \cdot \left(\frac{|y|}{|x|+|y|} \right) \leq |x| \rightarrow 0.$$

3.b $\lim_{(x,y) \rightarrow (0,0)} \frac{|x|(1+y)}{\sqrt{x^2+y^2}} = \text{non esiste}$

In $D = \mathbb{R}^2 \setminus \{(0,0)\}$, consideriamo le restrizioni lungo le rette passanti per $(0,0)$:

$$x=0, \quad f(0,y) = \frac{0}{|y|} = 0 \rightarrow 0 \text{ per } (x,y) \rightarrow (0,0)$$

$$y=mx, \quad f(x, mx) = \frac{|x|(1+mx)}{|x|\sqrt{1+m^2}} \rightarrow \frac{1}{\sqrt{1+m^2}} \neq 0.$$

Dato che i due limiti sono diversi si conclude che il limite dato non esiste.

3.c $\lim_{(x,y) \rightarrow (0,0)} \frac{x \log(1+x^2+2y^2)}{3x^2+y^2} = 0$

In $D = \mathbb{R}^2 \setminus \{(0,0)\}$, per $(x,y) \rightarrow (0,0)$ si ha che

$$\frac{x \log(1+x^2+2y^2)}{3x^2+y^2} = \left(\frac{\log(1+x^2+2y^2)}{x^2+2y^2} \right) \cdot \left(\frac{x \cdot (x^2+2y^2)}{3x^2+y^2} \right) \rightarrow 0$$

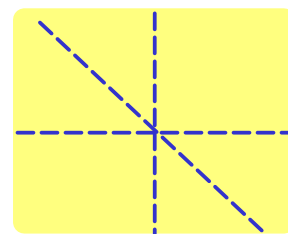
$\xrightarrow{1} \quad \xrightarrow{0}$

perché $\lim_{t \rightarrow 0} \frac{\log(1+t)}{t} = 1$ e

$$\left| \frac{x \cdot (x^2+2y^2)}{3x^2+y^2} \right| = |x| \cdot \left(\frac{x^2}{3x^2+y^2} \right) + 2|x| \cdot \left(\frac{y^2}{3x^2+y^2} \right) \leq 3|x| \rightarrow 0$$

$\leq 1 \quad \leq 1$

3.d $\lim_{(x,y) \rightarrow (0,0)} \frac{(x^2-y^2) \sin\left(\frac{1}{xy^2}\right)}{e^{x+y}-1} = 0$



In $D = \{(x,y) \in \mathbb{R}^2 : x \neq 0 \text{ e } y \neq 0 \text{ e } x+y \neq 0\}$

per $(x,y) \rightarrow (0,0)$ si ha che

$$\frac{(x^2-y^2) \sin\left(\frac{1}{xy^2}\right)}{e^{x+y}-1} = \left(\frac{x+y}{e^{x+y}-1} \right) \cdot \frac{(x+y)(x-y)}{x+y} \cdot \left(\sin\left(\frac{1}{xy^2}\right) \right) \rightarrow 0$$

$\xrightarrow{1} \quad \xrightarrow{0} \quad | \cdot | \leq 1$

perché $\lim_{t \rightarrow 0} \frac{e^t-1}{t} = 1$

3.e $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2(1-y^2)+y^4}{x^2+y^4} = 1$

Per $(x,y) \rightarrow (0,0)$,

$$\frac{x^2(1-y^2)+y^4}{x^2+y^4} = 1 - \left(\frac{x^2 y^2}{x^2+y^4} \right) \rightarrow 1$$

$\xrightarrow{0}$

perché

$$0 \leq \frac{x^2 y^2}{x^2+y^4} = \left(\frac{x^2}{x^2+y^4} \right) y^2 \leq y^2 \rightarrow 0$$

≤ 1

$$\boxed{3.f} \quad \lim_{(x,y) \rightarrow (0,0)} \frac{xy^2 \log(|x|/|y|)}{\sqrt{x^2+y^2}} = 0$$

Per $(x,y) \rightarrow (0,0)$ in $D = \mathbb{R}^2 \setminus \{(x,y) : x=0 \text{ o } y=0\}$

$$\frac{xy^2 \log(|x|/|y|)}{\sqrt{x^2+y^2}} = \left(\frac{y^2}{\sqrt{x^2+y^2}} \right) \underbrace{x \cdot \log(|x|)}_{\rightarrow 0} - \left(\frac{xy}{\sqrt{x^2+y^2}} \right) \underbrace{y \cdot \log(|y|)}_{\rightarrow 0}$$

Si ricorda che $\lim_{t \rightarrow 0} t^\alpha \log(t) = 0 \quad \forall \alpha > 0$. Inoltre

$$\alpha + \beta > 1 \text{ e } \alpha, \beta \geq 0 \Rightarrow 0 \leq \frac{|x^\alpha y^\beta|}{\sqrt{x^2+y^2}} \leq \rho^{(\alpha+\beta-1)} \xrightarrow{>0} 0.$$

$$\boxed{3.g} \quad \lim_{(x,y) \rightarrow \infty} \frac{x^4 + y^2 - 2x + 3y}{2 + \sqrt{x^2 + y^2}} = +\infty$$

Notiamo che $\pm x \leq |x| \leq \sqrt{x^2 + y^2}$ e $\pm y \leq |y| \leq \sqrt{x^2 + y^2}$.

Inoltre $(x^2 - 1)^2 \geq 0 \Rightarrow x^4 \geq 2x^2 - 1 \geq x^2 - 1$ e quindi

$$x^4 + y^2 - 2x + 3y \geq x^2 - 1 + y^2 - 2\sqrt{x^2 + y^2} - 3\sqrt{x^2 + y^2}$$

Infine per $\rho = \sqrt{x^2 + y^2} \rightarrow +\infty$,

$$\frac{x^4 + y^2 - 2x + 3y}{2 + \sqrt{x^2 + y^2}} \geq \frac{\rho^2 - 5\rho - 1}{2 + \rho} \rightarrow +\infty.$$

$$\boxed{3.h} \quad \lim_{(x,y) \rightarrow \infty} \frac{x^6 + 3y^3}{1 + x^2} = \text{non esiste}$$

Restringendo $f(x,y) = \frac{x^6 + 3y^3}{1 + x^2}$ lungo la retta $x=0$ si ha che

$$\lim_{y \rightarrow \pm\infty} f(0,y) = \lim_{y \rightarrow \pm\infty} 3y^3 = \pm\infty \quad \text{limiti diversi}$$

$$\boxed{3.i} \quad \lim_{(x,y) \rightarrow (1,1)} \frac{x^2 + xy - 2y^2}{x^2 - y^2} = \frac{3}{2}$$

Per $(x,y) \rightarrow (1,1)$ in $D = \mathbb{R}^2 \setminus \{(x,y) : |x| = |y|\}$

$$\frac{x^2 + xy - 2y^2}{x^2 - y^2} = 1 + \frac{xy - y^2}{x^2 - y^2} = 1 + \frac{y}{x+y} \rightarrow 1 + \frac{1}{2} = \frac{3}{2}.$$

$$\boxed{3.j} \quad \lim_{(x,y) \rightarrow (2,1)} \frac{x^2 - 4y}{x - 2y} = \text{non esiste}$$

Posto $u = x - 2$ e $v = y - 1$,

$$\begin{aligned} \lim_{(x,y) \rightarrow (2,1)} \frac{x^2 - 4y}{x - 2y} &= \lim_{(u,v) \rightarrow (0,0)} \frac{(u+2)^2 - 4(v+1)}{u+2 - 2(v+1)} \\ &= \lim_{(u,v) \rightarrow (0,0)} \frac{u^2 + 4u - 4v}{u - 2v} \end{aligned}$$

R restringendo $\neq$ al fascio di rette $u = mv$ con $m \neq 2$

$$\frac{u^2 + 4u - 4v}{u - 2v} = \frac{m^2v^2 + 4mv - 4v}{(m-2)v} \xrightarrow{v \rightarrow 0} \frac{4(m-1)}{m-2} \quad \text{limiti diversi}$$

Ad esempio per $m=0$ il limite è 2 mentre per $m=1$ il limite è 0.

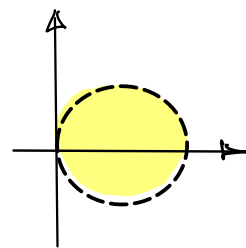
$$\boxed{3.k} \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x(1 - \cos(y^2(x+2)))}{x^4 + y^4} = 0$$

Dato che $t = y^2(x+2) \rightarrow 0$ e $\cos(t) = 1 - \frac{t^2}{2} + o(t^2)$

$$\begin{aligned} \frac{x(1 - \cos(y^2(x+2)))}{x^4 + y^4} &= \frac{x(y^2(x+2))^2 (\frac{1}{2} + o(1))}{x^4 + y^4} \\ &= \frac{\int_0^{2\pi} \cos \theta \cdot \int_0^1 m^4 \theta \cdot (4 + o(1)) (\frac{1}{2} + o(1))}{\int_0^{2\pi} (\underbrace{\cos^4 \theta + m^4 \theta}_{\geq \text{minimo positivo in } [0, 2\pi]})} = \int_0^{2\pi} (\dots) \xrightarrow{\rightarrow 0} 0 \end{aligned}$$

limitate

3.e $\lim_{(x,y) \rightarrow (0,0)} \frac{y}{\sqrt{2x-x^2-y^2}} = \text{non esiste}$



Si noti che $D = \{(x,y) : \frac{(x-1)^2 + y^2 < 1}{2x-x^2-y^2} > 0\}$.

Se restringiamo f lungo $y=0$ con $x>0$,

$$f(x,0) = 0 \rightarrow 0 \text{ per } x \rightarrow 0^+$$

mentre se restringiamo f lungo $y=x^\alpha$ con $\alpha>0$ e $x>0$, si nota che per $\alpha=\frac{1}{2}$ e $x \rightarrow 0^+$,

$$f(x, x^{\alpha}) = \frac{x^{\alpha}}{\sqrt{2x-x^2-x^{2\alpha}}} \stackrel{\alpha=\frac{1}{2}}{=} \frac{x^{\frac{1}{2}}}{\sqrt{2x-x^2-x}} \sim \frac{\sqrt{x}}{\sqrt{x}} \rightarrow 1 \neq 0. \quad \text{il limite non esiste}$$

3.m $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^3 + y^3} = \text{non esiste}$

Si noti che $D = \{(x,y) : y \neq -x\}$.

Se restringiamo f lungo $y=0$ con $x>0$,

$$f(x,0) = 0 \rightarrow 0 \text{ per } x \rightarrow 0^+$$

mentre se restringiamo f lungo $y=-x+x^\alpha$ con $\alpha>0$, $x>0$ e $\alpha>1$

$$\begin{aligned} f(x, -x+x^\alpha) &= \frac{x^2(-x+x^\alpha)^2}{x^3+(-x+x^\alpha)^3} = \frac{x^4 + o(x^4)}{\cancel{x^3} - \cancel{x^3} + 3x^{2+\alpha} + o(x^{2+\alpha})} \\ &= \frac{x^4 + o(x^4)}{3x^{2+\alpha} + o(x^{2+\alpha})} \rightarrow \begin{cases} 0 & \text{se } 1 < \alpha < 2 \\ 1/3 & \text{se } \alpha = 2 \\ +\infty & \text{se } \alpha > 2 \end{cases} \text{ per } x \rightarrow 0^+ \end{aligned}$$

Visto che per $\alpha \geq 2$ il limite è diverso da 0 il limite dato non esiste.

3.M $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{|x|^3 + |y|^3} = 0$

Si noti che $D = \mathbb{R}^2 \setminus \{(0,0)\}$.

$$f(x,y) \stackrel{CP}{=} \frac{\overset{\leq 1}{\cancel{\rho^4}} (\overset{\leq 1}{\cos \theta})^2 (\overset{\leq 1}{\sin \theta})^2}{\cancel{\rho^3} (\underbrace{|\cos \theta| + |\sin \theta|}_{\geq \text{minimo positivo in } [0, 2\pi]})^3} = \cancel{\rho} \cdot (\dots) \xrightarrow{\text{limitato}} 0 \text{ per } \rho \rightarrow 0.$$

3.O $\lim_{(x,y) \rightarrow (0,0)} \frac{x+y}{\sin(x)+y} = \text{non esiste}$

Si noti che $D = \{(x,y) : y \neq -\sin(x)\}$.

Per $x=0$ e $0 < y < 1$,

$$f(0,y) = \frac{y}{y} = 1 \rightarrow 1 \text{ per } y \rightarrow 0^+.$$

Se invece $y = -x + x^\alpha$ con $\alpha > 1$ per $x \rightarrow 0^+$,

$$\begin{aligned} f(x, -x + x^\alpha) &= \frac{x^\alpha}{\cancel{x} - \frac{x^3}{6} + o(x^3) - \cancel{x} + x^\alpha} \\ &\stackrel{\alpha=3}{=} \frac{x^3}{\frac{5}{6}x^3 + o(x^3)} \rightarrow \frac{6}{5} \neq 1 \end{aligned}$$

e quindi il limite dato non esiste.

3.P $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3(x+2y)}{x^4+y^4} = \text{non esiste}$

Lungo la retta $y = mx$ per $x \rightarrow 0^+$ si ha che

$$f(x, mx) = \frac{\cancel{x^4}(1+2m)}{\cancel{x^4}(1+m^4)} \rightarrow \frac{1+2m}{1+m^4} \begin{matrix} \xrightarrow{m=0} 1 \\ \searrow \\ \xrightarrow{m=1} \frac{3}{2} \end{matrix} \quad \text{il limite non esiste}$$

3.9

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2y) - x \sin(xy)}{x^4y^3} = \frac{1}{6}$$

Per $(x,y) \rightarrow (0,0)$,

$$\sin(t) = t - \frac{t^3}{6} + o(t^3) \text{ per } t \rightarrow 0$$

$$\sin(x^2y) = x^2y - \frac{1}{6}x^6y^3 + o(x^6y^3)$$

$$\begin{aligned} x \sin(xy) &= x \left(xy - \frac{1}{6}x^3y^3 + o(x^3y^3) \right) \\ &= x^2y - \frac{1}{6}x^4y^3 + o(x^4y^3) \end{aligned}$$

e quindi

$$\begin{aligned} \frac{\sin(x^2y) - x \sin(xy)}{x^4y^3} &= \frac{\cancel{x^2y} - \frac{1}{6}x^6y^3 + o(x^6y^3) - \cancel{x^2y} + \frac{1}{6}x^4y^3 + o(x^4y^3)}{x^4y^3} \\ &= -\frac{x^2}{6} + o(x^2) + \frac{1}{6} + o(1) \rightarrow \frac{1}{6}. \end{aligned}$$

3.10

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 \log(1-y) + y \log(1+x^2)}{x^2y^2} = \text{non esiste}$$

Per $(x,y) \rightarrow (0,0)$,

$$\log(1+t) = t - \frac{t^2}{2} + o(t^2) \text{ per } t \rightarrow 0$$

$$x^2 \log(1-y) + y \log(1+x^2) = x^2 \left(-y - \frac{(-y)^2}{2} + o(y^2) \right)$$

$$+ y \left(x^2 - \frac{x^4}{2} + o(x^4) \right)$$

$$= -\frac{x^2y^2}{2} + o(x^2y^2) - \frac{x^4y}{2} + o(x^4y).$$

(non confrontabili)

Si noti che se $y=x$, per $x \rightarrow 0^+$

$$f(x,x) = \frac{-\frac{x^4}{2} + o(x^4) - \frac{x^5}{2} + o(x^5)}{x^4} \rightarrow -\frac{1}{2}$$

invece se $y=x^2$, per $x \rightarrow 0^+$

$$f(x,x^2) = \frac{-\frac{x^6}{2} + o(x^6) - \frac{x^6}{2} + o(x^6)}{x^6} \rightarrow -1$$

il limite non esiste