ANALISI MATEMATICA 2 - LEZIONE 40

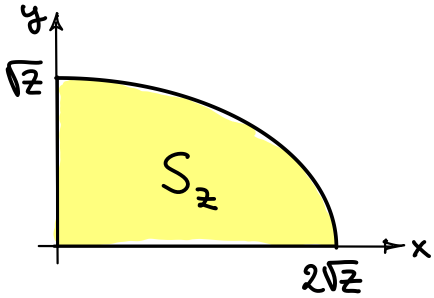
ESERCIZI DI RIEPILOGO - PARTE 3

$\boxed{11}$ Calcolare $\displaystyle\lim_{r\to 0^+} \iiint_{D_r} xy\frac{\log(z)}{z^2}\,dx\,dy\,dz$

con $D_r = \{(x,y,z) : x \geq 0,\ y \geq 0,\ r \leq z \leq 2,\ x^2+4y^2 \leq 4z\}$.

Integriamo per sezioni con $r \leq z \leq 2$

$$S_z = \{(x,y) : \underbrace{x^2+4y^2 \leq 4z}_{\frac{x^2}{(2\sqrt{z})^2}+\frac{y^2}{(\sqrt{z})^2}\leq 1},\ x\geq 0,\ y \geq 0\}$$



Allora

$$\iiint_{D_r} xy\frac{\log(z)}{z^2}\,dx\,dy\,dz = \int_{z=r}^{2}\frac{\log(z)}{z^2}\left(\iint_{S_z} xy\,dx\,dy\right)dz$$

$$\begin{cases} x = 2\rho\cos\theta \\ y = \rho\sin\theta \end{cases} \quad \vec{\Phi}(\rho,\theta)$$

$$= \int_{z=r}^{2}\frac{\log(z)}{z^2}\int_{\rho=0}^{\sqrt{z}}\int_{\theta=0}^{\frac{\pi}{2}} 2\rho\cos\theta\cdot\rho\sin\theta\cdot\underbrace{(2\rho)}_{|\det J_{\Phi}|}\,d\rho\,d\theta$$

$$= 4\int_{z=r}^{2}\frac{\log(z)}{z^2}\left[\frac{\rho^4}{4}\right]_0^{\sqrt{z}}\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}}dz$$

$$= \frac{1}{2}\int_{z=r}^{2}\frac{\cancel{z^2}\log(z)}{\cancel{z^2}}\,dz = \frac{1}{2}\Big[z\log(z)-z\Big]_r^2$$

$$= \log(2)-1-\frac{1}{2}(r\log(r)-r) \xrightarrow{r\to 0^+} \log(2)-1.$$

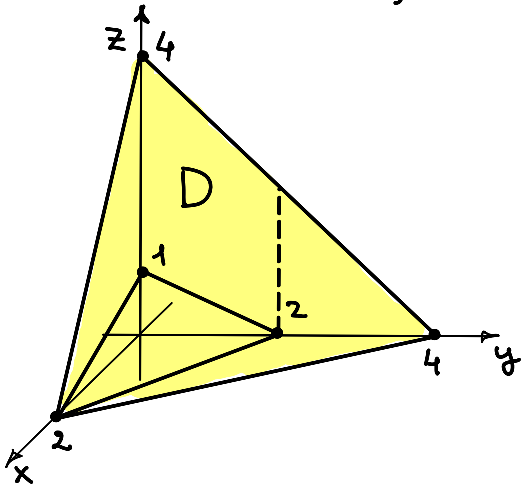
$\boxed{12}$ Calcolare $\iiint_D |x-1|\,dx\,dy\,dz$ con

$$D=\{(x,y,z):\ x+y+2z\geq 2,\ 2x+y+z\leq 4,\ x\geq 0,\ y\geq 0,\ z\geq 0\}.$$

D è il poliedro di vertici:

$(2,0,0), (0,4,0), (0,0,4)$ sul piano $2x+y+z=4$

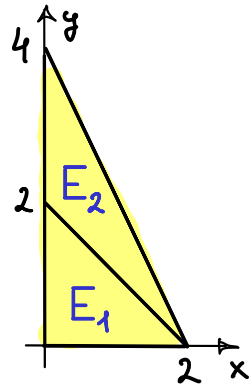
$(2,0,0), (0,2,0), (0,0,1)$ sul piano $x+y+2z=2$



Integrazione per fili:

$$\iiint_D |x-1|\,dx\,dy\,dz=\int_{x=0}^{2}|x-1|\left(\int_{y=0}^{2-x}\left(\int_{z=1-\frac{x+y}{2}}^{4-2x-y}dz\right)dy\right)dx \quad E_1$$

$$+\int_{x=0}^{2}|x-1|\left(\int_{y=2-x}^{4-2x}\left(\int_{z=0}^{4-2x-y}dz\right)dy\right)dx \quad E_2$$



$$=\dots=\int_0^2|x-1|\left(5-5x+\frac{5}{4}x^2\right)dx+\int_0^2|x-1|\left(2-2x+\frac{1}{2}x^2\right)dx$$

$$=\int_0^2|x-1|\underbrace{\left(7-7x+\frac{7}{4}x^2\right)}_{\frac{7}{4}(4-4x+x^2)}dx=\frac{7}{4}\int_0^2|x-1|(x-2)^2dx$$

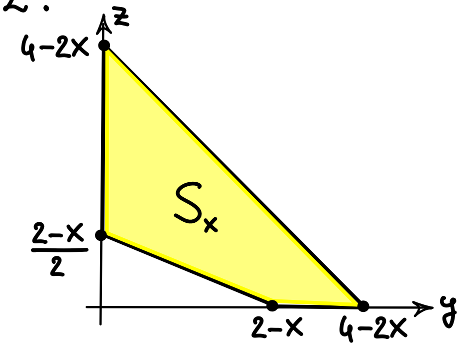
$t=x-1$

$$=\frac{7}{4}\int_{-1}^{1}\underbrace{|t|}_{P}\underbrace{(t-1)^2}_{t^2-2t+1}dt=\frac{7}{\cancel{4}_2}\cdot\cancel{2}\int_0^1(t^3+t)\,dt$$

P d P

$$=\frac{7}{2}\left[\frac{t^4}{4}+\frac{t^2}{2}\right]_0^1=\frac{21}{8}.$$

Integrazione per sezioni con $0 \le x \le 2$:



$$S_x = \{(y,z) : y + 2z \geq 2 - x,\ y + z \leq 4 - 2x,\ y \geq 0,\ z \geq 0\}$$

$$\iiint_D |x-1|\,dx\,dy\,dz = \int_{x=0}^{2} |x-1| \left(\iint_{S_x} dy\,dz \right) dx$$

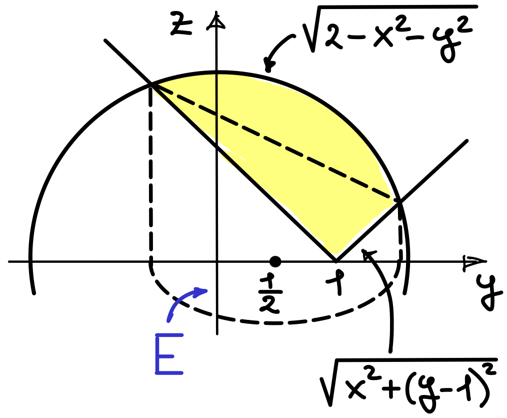
$$= \int_{x=0}^{2} |x-1| \cdot |S_x|\,dx = \int_{x=0}^{2} |x-1| \underbrace{\left(\frac{(4-2x)^2}{2} - \frac{(2-x)^2}{4} \right)}_{\frac{7}{4}(x-2)^2} dx$$

$$= \frac{7}{4} \int_0^2 |x-1|(x-2)^2\,dx = \dots = \frac{21}{8}.$$

$\boxed{13}$ Calcolare $\iiint_D z\,dx\,dy\,dz$

con $D = \{(x,y,z) : x \geq 0,\ x^2+y^2+z^2 \leq 2,\ z \geq \sqrt{x^2+(y-1)^2}\}$.

Integrazione per fili:



$$\sqrt{x^2+(y-1)^2} \leq z \leq \sqrt{2-x^2-y^2}$$

con $(x,y) \in E$ ossia $x \geq 0$ e

$$x^2+y^2+(x^2+(y-1)^2) \leq 2$$

$$2x^2+2y^2-2y+1 \leq 2$$

$$x^2+y^2-y \leq \frac{1}{2}$$

$$x^2+\left(y-\frac{1}{2}\right)^2 \leq \frac{1}{4}+\frac{1}{2} = \frac{3}{4}.$$

Quindi E è il semicerchio di centro $(0,\frac{1}{2})$ e raggio $\frac{\sqrt{3}}{2}$ contenuto nel semipiano $x \geq 0$.

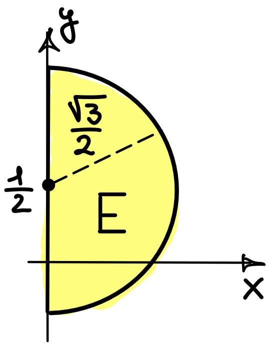
Allora

$$\iiint_D z\,dx\,dy\,dz = \iint_E \left(\int_{\sqrt{x^2+(y-1)^2}}^{\sqrt{2-x^2-y^2}} z\,dz \right) dx\,dy$$

$$= \frac{1}{2}\iint_E \left(\left(2-x^2-y^2\right)-\left(x^2+(y-1)^2\right)\right)dx\,dy$$

$\hookrightarrow 1-2x^2-2y^2+2y = 1-2x^2-2\left(y-\frac{1}{2}\right)^2+\frac{1}{2}$

$$= \iint_E \left(\frac{3}{4}-x^2-\left(y-\frac{1}{2}\right)^2\right)dx\,dy$$



$\begin{cases} x=\rho\cos\theta \\ y=\frac{1}{2}+\rho\sin\theta \end{cases}$

$$= \int_{\rho=0}^{\frac{\sqrt{3}}{2}}\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{3}{4}-\rho^2\right)\rho\,d\rho\,d\theta$$

$$=\pi\left[\frac{3}{4}\cdot\frac{\rho^2}{2}-\frac{\rho^4}{4}\right]_0^{\frac{\sqrt{3}}{2}} = \pi\left(\frac{3}{4}\right)^2\cdot\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{9\pi}{64}.$$

14 Calcolare la coordinata z del baricentro della curva

$$\vec{\gamma}(t)=\left(\frac{t^3}{3}+\frac{1}{t},\sqrt{3}t,t\right) \text{ con } t\in[1,2].$$

Si ha che $\vec{\gamma}'(t)=\left(t^2-\frac{1}{t^2},\sqrt{3},1\right)$ e quindi

$$\|\vec{\gamma}'(t)\|=\left(\left(t^2-\frac{1}{t^2}\right)^2+3+1\right)^{1/2}=\left(t^4-2+\frac{1}{t^4}+4\right)^{1/2}=t^2-\frac{1}{t^2}.$$

$\hookrightarrow t^4+2+\frac{1}{t^4}=\left(t^2+\frac{1}{t^2}\right)^2$

Lunghezza di γ:

$$|\gamma|=\int_\gamma ds=\int_1^2\left(t^2+\frac{1}{t^2}\right)dt=\left[\frac{t^3}{3}-\frac{1}{t}\right]_1^2=\frac{8}{3}-\frac{1}{2}-\frac{1}{3}+1=\frac{17}{6}.$$

Coordinata z del baricentro:

$$\bar{z}=\frac{1}{|\gamma|}\int_\gamma z\,ds=\frac{6}{17}\int_1^2 t\left(t^2+\frac{1}{t^2}\right)dt$$

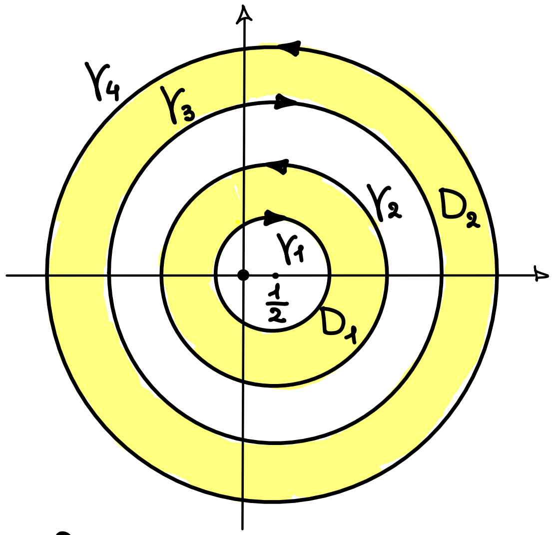
$$=\frac{6}{17}\left[\frac{t^4}{4}+\log|t|\right]_1^2=\frac{6}{17}\left(\frac{16-1}{4}+\log(2)\right)$$

$$=\frac{45}{34}+\frac{6}{17}\log(2).$$

$\boxed{15}$ Calcolare $\displaystyle\sum_{i=1}^{4}\int_{\gamma_i}\langle\vec{F},d\vec{s}\rangle$ dove

$$\vec{F}(x,y)=\left(-y\left(1+\frac{1}{x^2+y^2}\right),\, x\left(x+\frac{1}{x^2+y^2}\right)\right)$$

e $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ sono le circonferenze di centro $(\frac{1}{2},0)$ e raggi rispettivamente $1,2,3,4$.
γ_1 e γ_3 sono percorse in senso orario mentre γ_2 e γ_4 sono percorse in senso antiorario.



Osserviamo che $\vec{F}=\vec{F}_1+\vec{F}_2$ dove

$\vec{F}_1=(-y,x^2)$ e $\vec{F}_2=\left(\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2}\right)$.

$\vec{F}_2$ è irrotazionale in $\mathbb{R}^2\setminus\{(0,0)\}$

In alternativa al calcolo diretto applichiamo la formula di Gauss-Green a D_1 e D_2 indicati in figura.

$$\frac{\partial}{\partial x}\left(x+\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(-y-\frac{y}{x^2+y^2}\right)\overset{\vec{F}_2 \text{ è irrotazionale}}{=}\frac{\partial(x^2)}{\partial x}-\frac{\partial(-y)}{\partial y}=2x+1.$$

Quindi, dato che il baricentro di D_1 è $(\frac{1}{2},0)$,

$$\sum_{i=1}^{2}\int_{\gamma_i}\langle\vec{F},d\vec{s}\rangle=\iint_{D_1}(2x+1)\,dx\,dy=(2\bar{x}+1)|D_1|$$
$$=\left(2\cdot\frac{1}{2}+1\right)\cdot\left(\pi 2^2-\pi 1^2\right)=6\pi.$$

Analogamente per D_2

$$\sum_{i=3}^{4}\int_{\gamma_i}\langle\vec{F},d\vec{s}\rangle=\iint_{D_2}(2x+1)\,dx\,dy=(2\bar{x}+1)|D_2|$$
$$=\left(2\cdot\frac{1}{2}+1\right)\cdot\left(\pi 4^2-\pi 3^2\right)=14\pi.$$

Così il risultato finale è

$$\sum_{i=1}^{4}\int_{\gamma_i}\langle \vec{F}, d\vec{s}\rangle = \sum_{i=1}^{2}\int_{\gamma_i}\langle \vec{F}, d\vec{s}\rangle + \sum_{i=3}^{4}\int_{\gamma_i}\langle \vec{F}, d\vec{s}\rangle$$

$$= 6\pi + 14\pi = 20\pi .$$

OSSERVAZIONE

Come visto a lezione, per $\vec{F_2} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$, dato che ogni γ_i si avvolge intorno a $(0,0)$, si ha che

$$\int_{\gamma_1}\langle \vec{F_2}, d\vec{s}\rangle = \int_{\gamma_3}\langle \vec{F_2}, d\vec{s}\rangle = -2\pi$$

e

$$\int_{\gamma_2}\langle \vec{F_2}, d\vec{s}\rangle = \int_{\gamma_4}\langle \vec{F_2}, d\vec{s}\rangle = +2\pi .$$