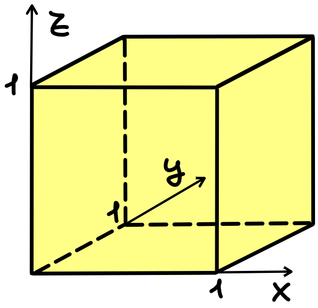
ANALISI MATEMATICA 2 - LEZIONE 37

ALCUNI ESERCIZI DEL FOGLIO 8

1.a Calcolare $\iint_S (x^2+z^2)\,dS$ dove S è la superficie del cubo $[0,1]^3$.



Per simmetria $\iint_S x^2\,dS = \iint_S z^2\,dS$.

$S=\bigcup_{i=1}^{6} S_i$ è regolare a pezzi:

$$S_1=\{(x,y,0): x,y\in[0,1]\}\ ,\ S_2=\{(x,y,1): x,y\in[0,1]\}$$

e quindi

$$\iint_{S_1} x^2\,dS = \iint_{S_2} x^2\,dS = \int_0^1\!\!\int_0^1 x^2\,dx\,dy = \left[\frac{x^3}{3}\right]_0^1 \left[y\right]_0^1 = \frac{1}{3}.$$

Inoltre

$$S_3=\{(x,0,z): x,z\in[0,1]\}\ ,\ S_4=\{(x,1,z): x,z\in[0,1]\},$$

$$S_5=\{(0,y,z): y,z\in[0,1]\}\ ,\ S_6=\{(1,y,z): y,z\in[0,1]\}$$

e così

$$\iint_{S_3} x^2\,dS = \iint_{S_4} x^2\,dS = \int_0^1\!\!\int_0^1 x^2\,dx\,dz = \left[\frac{x^3}{3}\right]_0^1 \left[z\right]_0^1 = \frac{1}{3},$$

$$\iint_{S_5} \underbrace{x^2}_{=0}\,dS = 0\ ,\ \iint_{S_6} \underbrace{x^2}_{=1}\,dS = |S_6| = 1.$$

Infine

$$\iint_S (x^2+z^2)\,dS = 2\sum_{i=1}^{6}\iint_{S_i} x^2\,dS = 2\left(4\cdot\frac{1}{3}+0+1\right) = \frac{14}{3}.$$

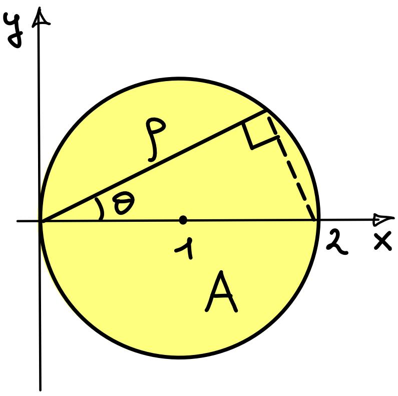
$\boxed{1.b}$ Calcolare $\iint_S x\sqrt{z^2-4x^2}\,dS$ dove S è

$$S=\{(x,y,z):\ \underbrace{z=2\sqrt{x^2+y^2}}_{\text{cono}},\ \underbrace{(x-1)^2+y^2\le 1}_{\text{cilindro}}\}.$$

Parametrizzazione di S:

$$\vec{\sigma}(x,y)=(x,y,2\sqrt{x^2+y^2})$$

con $A=\{(x,y):\ (x-1)^2+y^2\le 1\}$.



Così

$$\vec{\sigma}_x\times\vec{\sigma}_y=\left(-\frac{2x}{\sqrt{x^2+y^2}},-\frac{2y}{\sqrt{x^2+y^2}},1\right).$$

Quindi $\|\vec{\sigma}_x\times\vec{\sigma}_y\|=\sqrt{5}$ e

$$\iint_S x\sqrt{z^2-4x^2}\,dS=\iint_A x\underbrace{\sqrt{\cancel{4x^2}+4y^2-\cancel{4x^2}}}_{=2|y|}\cdot\sqrt{5}\,dx\,dy$$

$$\overset{CP}{=}2\sqrt{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}\rho\cos\theta\cdot\rho|\sin\theta|\,\rho\,d\rho\,d\theta$$

$$\overset{CP}{=}2\sqrt{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\cdot|\sin\theta|\left[\frac{\rho^4}{4}\right]_0^{2\cos\theta}d\theta$$

$$=8\sqrt{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\overbrace{\cos^5\theta\cdot|\sin\theta|}^{\theta\text{-pari}}\,d\theta=16\sqrt{5}\int_0^{\frac{\pi}{2}}\cos^5\theta\,d(-\cos\theta)$$

$$=16\sqrt{5}\left[-\frac{\cos^6\theta}{6}\right]_0^{\frac{\pi}{2}}=\frac{8\sqrt{5}}{3}.$$

2.a Calcolare il flusso $\iint_S \langle \vec{F}, d\vec{S} \rangle$ dove $\vec{F} = (3e^{x+y}, 2y-z, xy)$, $S = \partial D$ e

$$D = \{(x,y,z) : |x-1| + |y| \leq 1,\ 0 \leq z \leq x+y\}.$$

S è orientata verso l'esterno.

Applichiamo il teorema della divergenza:

$$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(3e^{x+y}) + \frac{\partial}{\partial y}(2y-z) + \frac{\partial}{\partial z}(xy) = 3e^{x+y} + 2$$

e allora

$$\iint_S \langle \vec{F}, d\vec{S} \rangle \overset{TD}{=} \iiint_D \operatorname{div}(\vec{F})\,dx\,dy\,dz$$

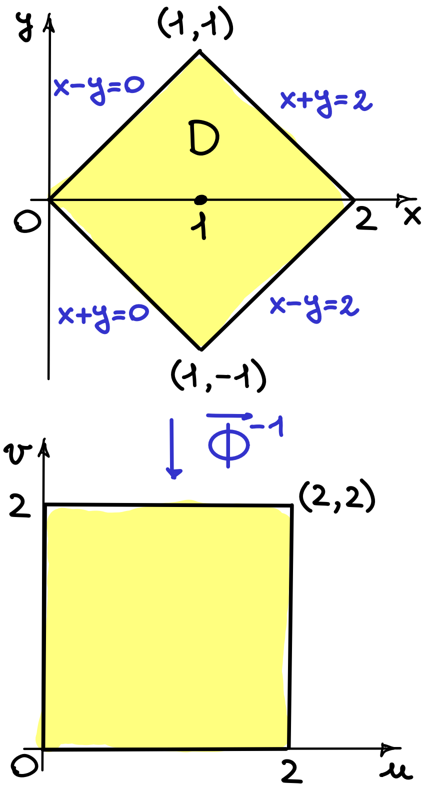
$$= \iint_{\{|x-1|+|y|\leq 1\}} (3e^{x+y}+2)\left(\int_{z=0}^{x+y} dz\right) dx\,dy$$

$$= \iint_{\{|x-1|+|y|\leq 1\}} (3e^{x+y}+2)(x+y)\,dx\,dy$$

$$\begin{cases} u = x+y \\ v = x-y \end{cases} \ \vec{\Phi}^{-1} \quad \overset{\curvearrowright}{=} \iint_{[0,2]^2 = \vec{\Phi}^{-1}(\{|x-1|+|y|\leq 1\})} (3e^u+2)\,u \underbrace{\left(\frac{1}{2}\right)}_{= |\det J_{\vec{\Phi}}|} du\,dv$$

$$= \int_0^2 (3e^u u + 2u)\,du = \Big[3e^u(u-1) + u^2\Big]_0^2$$

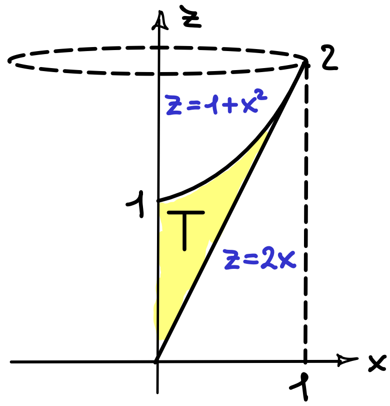
$$= 3e^2 + 4 - (-3) = 3e^2 + 7.$$



2.b Calcolare il flusso $\iint_S \langle \vec{F}, d\vec{S} \rangle$ dove $\vec{F} = (e^{y^2}, 5x^2y, 5y^2z)$, $S = \partial D$ è orientata verso l'esterno e D è generato dalla rotazione completa di

$$T = \{(x,0,z) : 0 \leq 2x \leq z \leq 1 + x^2 \leq 2\}$$

intorno all'asse z.



Applichiamo il teorema della divergenza:

$$\mathrm{div}(\vec{F}) = \frac{\partial}{\partial x}(e^{y^2}) + \frac{\partial}{\partial y}(5x^2y) + \frac{\partial}{\partial z}(5y^2z) = 5(x^2+y^2).$$

e allora

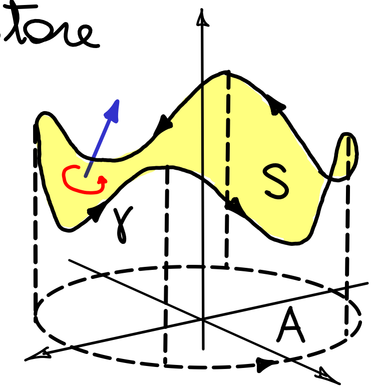
$$\iint_S \langle \vec{F}, d\vec{S} \rangle \overset{TD}{=} \iiint_D \mathrm{div}(\vec{F})\,dx\,dy\,dz$$

$$\overset{CC}{=} \int_{\rho=0}^{1} 5\rho^2 \left(\int_{z=2\rho}^{1+\rho^2} \left(\int_{\theta=0}^{2\pi} d\theta \right) dz \right) \rho\,d\rho = 10\pi \int_0^1 \rho^3(1+\rho^2-2\rho)\,d\rho$$

$$= 10\pi \left[\frac{\rho^4}{4} + \frac{\rho^6}{6} - 2\frac{\rho^5}{5} \right]_0^1 = 10\pi \cdot \frac{15+10-24}{60} = \frac{\pi}{6}.$$

3.a

Usando il teorema del rotore calcolare $\int_\gamma \langle \vec{F}, d\vec{s}\rangle$ dove



$\vec{F} = (\operatorname{sen} x, z(x-1), y(x+1))$ e

$\gamma = \{(x,y,z) : x^2+y^2=4,\ z=2+x^2y^2\}$.

γ è orientata in modo che la sua proiezione su $z=0$ sia in verso antiorario.

Sia $S = \{(x,y,z) : x^2+y^2 \le 4,\ z = 2+x^2y^2\}$.

Parametrizzazione cartesiana di S:

$$\vec{\sigma}(x,y) = (x, y, 2+x^2y^2) \text{ con } A = \{(x,y) : x^2+y^2 \le 4\}.$$

Allora $\vec{\sigma}_x \times \vec{\sigma}_y = (-2xy^2, -2yx^2, 1)$ e $\vec{\sigma}$ induce su $\gamma = \partial^+ S$ l'orientazione richiesta. Calcolo del rotore:

$$\operatorname{rot}(\vec{F}) = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \operatorname{sen} x & z(x-1) & y(x+1)\end{bmatrix} = (2, -y, z).$$

Quindi

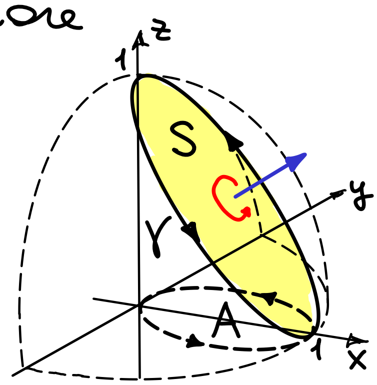
$$\int_\gamma \langle \vec{F}, d\vec{s}\rangle \overset{TR}{=} \iint_S \langle \operatorname{rot}(\vec{F}), d\vec{S}\rangle$$

$$= \iint_A \langle (2, -y, \underset{\overset{\shortparallel}{2+x^2y^2}}{z}), (-2xy^2, -2yx^2, 1)\rangle\, dx\,dy$$

$$= \iint_{\{x^2+y^2\le 4\}} (\underbrace{-4xy^2}_{x\text{-dispari}} + 3x^2y^2 + 2)\,dx\,dy \overset{CP}{=} 3\int_0^{2\pi}\int_0^2 \rho^4 \underbrace{\cos^2\theta \cdot \operatorname{sen}^2\theta}_{\frac{1}{4}\operatorname{sen}^2 2\theta}\, \rho\, d\rho\, d\theta + 2\underset{\overset{\shortparallel}{4\pi}}{|A|}$$

$$= \frac{\cancel{3}}{4}\int_0^{2\pi} \operatorname{sen}^2 2\theta\, d\theta \left[\frac{\rho^6}{\cancel{6}_{2}}\right]_0^2 + 8\pi = \frac{1}{4}\pi \cdot 32 + 8\pi = 16\pi.$$

3.b Usando il teorema del rotore calcolare $\int_\gamma \langle \vec{F}, d\vec{s} \rangle$ dove



$\vec{F} = (y + ye^{xy}, xe^{xy}, x^3)$ e

$\gamma = \{(x,y,z) : x^2+y^2+z^2=1, x+z=1\}$

orientata in modo che la sua proiezione su $z=0$ sia in verso antiorario.

Sia $S = \{(x,y,z) : x^2+y^2+z^2 \le 1, x+z=1\}$.

Parametrizzazione cartesiana di S:

$\vec{\sigma}(x,y) = (x, y, 1-x)$ con $A = \{(x,y) : x^2+y^2+(1-x)^2 \le 1\}$.

Allora $\vec{\sigma}_x \times \vec{\sigma}_y = (1,0,1)$ e $\vec{\sigma}$ induce su $\gamma = \partial^+ S$ l'orientazione richiesta. Calcolo del rotore:

$$\mathrm{rot}(\vec{F}) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y+ye^{xy} & xe^{xy} & x^3 \end{bmatrix} = (0, -3x^2, -1).$$

Quindi

$$\int_\gamma \langle \vec{F}, d\vec{s} \rangle \overset{TR}{=} \iint_S \langle \mathrm{rot}(\vec{F}), d\vec{S} \rangle$$

$$= \iint_A \langle (0,-3x^2,-1), (1,0,1) \rangle \, dx\,dy$$

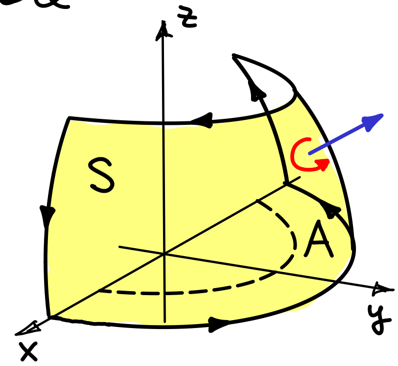
$$= -\iint_A dx\,dy = -|A| = -\pi \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = -\frac{\pi}{2\sqrt{2}}.$$

∂A è un'ellisse di semiassi $\frac{1}{2}$ e $\frac{1}{\sqrt{2}}$.

$x^2+y^2+(1-x)^2 = 1 \iff 2x^2-2x+y^2=0$

$\iff 2\left(x-\frac{1}{2}\right)^2 + y^2 = \frac{1}{2} \iff \frac{\left(x-\frac{1}{2}\right)^2}{(1/2)^2} + \frac{y^2}{(1/\sqrt{2})^2} = 1$

$\boxed{4.a}$ Usando il teorema del rotore calcolare $\int_{\partial^+ S} \langle \vec{F}, d\vec{s} \rangle$ dove



$\vec{F} = (e^x, xy + 2x, x^2 + z^2)$ e

$S = \{(x,y,z) : x^2 + y^2 = 4 - z, 0 \leq z \leq 3, y \geq 0\}$

orientata in modo che $\langle \vec{n}, \vec{k} \rangle \geq 0$.

Parametrizzazione cartesiana di S:

$\vec{\sigma}(x,y) = (x, y, 4 - x^2 - y^2)$ con $A = \{(x,y) : 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$.

Allora $\vec{\sigma}_x \times \vec{\sigma}_y = (2x, 2y, \overset{>0}{1})$ e $\vec{\sigma}$ induce su $\gamma = \partial^+ S$ l'orientazione richiesta. Calcolo del rotore:

$$\operatorname{rot}(\vec{F}) = \det \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & xy + 2x & x^2 + z^2 \end{bmatrix} = (0, -2x, y + 2).$$

Quindi

$$\int_\gamma \langle \vec{F}, d\vec{s} \rangle \overset{TR}{=} \iint_S \langle \operatorname{rot}(\vec{F}), d\vec{S} \rangle$$

$$= \iint_A \langle (0, -2x, y+2), (2x, 2y, 1) \rangle \, dx\,dy$$

$$= \iint_A (-\underbrace{4xy}_{x\text{-dispari}} + y + 2) \, dx\,dy$$

simmetrico per $x = 0$ $\to$ A

$$\overset{CP}{=} \int_{\theta=0}^{\pi} \int_{\rho=1}^{2} \rho \sin\theta \, \rho \, d\rho \, d\theta + 2 \underbrace{|A|}_{(4\pi - \pi)/2}$$

$$= \left[-\cos\theta\right]_0^\pi \cdot \left[\frac{\rho^3}{3}\right]_1^2 + 3\pi = 2\,\frac{8-1}{3} + 3\pi = \frac{14}{3} + 3\pi.$$