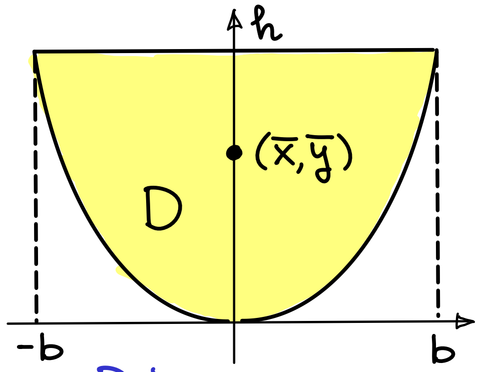
ANALISI MATEMATICA 2 - LEZIONE 27

ALCUNI ESERCIZI DEL FOGLIO 6

1.a Calcolare il baricentro di

$$D=\left\{(x,y):\ \frac{hx^2}{b^2}\le y\le h\right\}$$

con $h,b>0$.



D è un segmento parabolico

Per simmetria

$$\overline{x}=\frac{1}{|D|}\iint_D x\,dx\,dy=0$$

Calcolo di $|D|$:

$$|D|=\iint_D 1\,dx\,dy=2\int_{x=0}^{b}\left(\int_{y=\frac{hx^2}{b^2}}^{h}dy\right)dx=2\int_0^b\left(h-\frac{hx^2}{b^2}\right)dx$$

$$=2h\left[x-\frac{x^3}{3b^2}\right]_0^b=2h\left(b-\frac{b}{3}\right)=\frac{4hb}{3}.$$

Calcolo di $\overline{y}$:

$$\overline{y}=\frac{1}{|D|}\iint_D y\,dx\,dy=\frac{2}{|D|}\int_{x=0}^{b}\left(\int_{y=\frac{hx^2}{b^2}}^{h}y\,dy\right)dx$$

$$=\frac{1}{|D|}\int_0^b\left[y^2\right]_{\frac{hx^2}{b^2}}^{h}dx=\frac{3}{4hb}\int_0^b\left(h^2-\frac{h^2x^4}{b^4}\right)dx$$

$$=\frac{3h^{\cancel{2}}}{4\cancel{h}b}\left[x-\frac{x^5}{5b^4}\right]_0^b=\frac{3h}{4\cancel{b}}\left(\cancel{b}-\frac{\cancel{b}}{5}\right)$$

$$=\frac{3h}{\cancel{4}}\cdot\frac{\cancel{4}}{5}=\frac{3h}{5}.$$

1.d

$$\lim_{R\to+\infty}\iint_{D_R}\frac{1}{(1+x^2+y^2)^\alpha}dxdy \quad \text{per } \alpha>0$$

dove $D_R=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le R^2\}$.

$$\iint_{D_R}\frac{1}{(1+x^2+y^2)^\alpha}dxdy \overset{CP}{=} \int_{\theta=0}^{2\pi}\int_{\rho=0}^{R}\frac{1}{(1+\rho^2)^\alpha}\rho\, d\rho\, d\theta$$

$t=1+\rho^2,\ dt=2\rho d\rho$

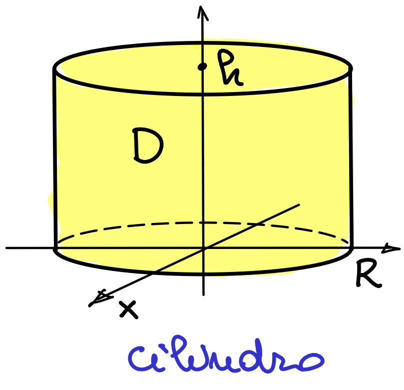
$$= \frac{\cancel{2}\pi}{\cancel{2}}\int_1^{1+R^2}\frac{1}{t^\alpha}dt = \begin{cases}\pi\left[\dfrac{t^{-\alpha+1}}{-\alpha+1}\right]_1^{1+R^2} & \text{se } \alpha\neq 1\\[2ex] \pi\left[\log(t)\right]_1^{1+R^2} & \text{se } \alpha=1\end{cases}$$

Quindi

$$\lim_{R\to+\infty}\iint_{D_R}\frac{1}{(1+x^2+y^2)^\alpha}dxdy = \begin{cases}\dfrac{\pi}{\alpha-1} & \text{se } \alpha>1\\[2ex] +\infty & \text{se } \alpha\le 1\end{cases}$$

1.h Calcolare I/M per $D=\left\{(x,y,z) : \begin{array}{l}x^2+y^2\le R^2\\ 0\le z\le h\end{array}\right\}$

con $h,R>0$ rispetto all'asse x.



La distanza di un punto (x,y,z) dall'asse x è $\sqrt{y^2+z^2}$.

Quindi

$$\frac{I}{M}=\frac{1}{|D|}\iiint_D (y^2+z^2)\,dxdydz$$

$$\overset{CC}{=}\frac{1}{\pi R^2 h}\int_{\rho=0}^{R}\int_{\theta=0}^{2\pi}\int_{z=0}^{h}(\rho^2\sin^2\theta+z^2)\rho\, d\rho\, d\theta\, dz$$

$$= \frac{1}{\pi R^2 h} \int_{\rho=0}^{R} \int_{\theta=0}^{2\pi} \left[\rho^3 \sin^2\theta \cdot z + \rho \frac{z^3}{3} \right]_0^h d\rho\, d\theta$$

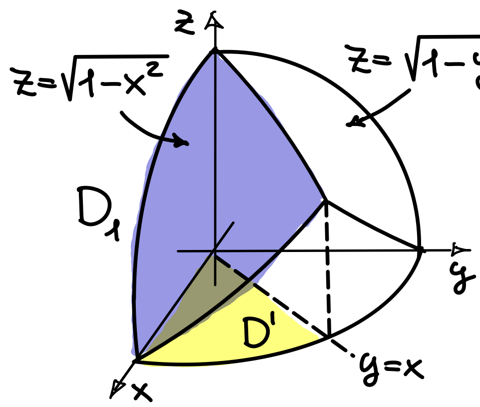
$$= \frac{\cancel{h}}{\pi R^2 \cancel{h}} \int_{\rho=0}^{R} \rho^3 d\rho \cdot \int_{\theta=0}^{2\pi} \sin^2\theta\, d\theta + \frac{h^{\cancel{3}\,2}}{3\pi R^2 \cancel{h}} \int_{\rho=0}^{R} \rho\, d\rho \cdot \int_{\theta=0}^{2\pi} d\theta$$

$$= \frac{1}{\cancel{\pi} \cancel{R^2}} \cdot \frac{R^{\cancel{4}\,2}}{4} \cdot \cancel{\pi} + \frac{h^2}{3\cancel{\pi}\cancel{R^2}} \cdot \frac{\cancel{R^2}}{\cancel{2}} \cdot \cancel{2\pi} = \frac{R^2}{4} + \frac{h^2}{3}.$$

$\boxed{1.j}$ Calcolare il volume di

$$D = \{(x,y,z) : x^2+y^2 \le 1,\ y^2+z^2 \le 1,\ x^2+z^2 \le 1\}.$$

$$(x,y,z) \in D \iff \begin{cases} x^2+y^2 \le 1 \text{ e} \\ |z| \le \min\left(\sqrt{1-x^2}, \sqrt{1-y^2}\right) \\ \qquad \sqrt{1-x^2} \text{ se } |y| \le |x| \end{cases}$$



Dato che D è simmetrico rispetto ai piani $x=0$, $y=0$, $z=0$ e $y=x$.

$|D| = 8 \cdot 2 |D_1|$ dove

$$D_1 = \{(x,y,z) : (x,y) \in D',\ 0 \le z \le \sqrt{1-x^2}\} \text{ e}$$

$$D' = \{(x,y) : x \in [0,1],\ 0 \le y \le x,\ 0 \le y \le \sqrt{1-x^2}\}.$$

Calcolo di $|D_1|$:

$$|D_1| = \iint_{D'} \sqrt{1-x^2}\, dx\, dy \overset{CP}{=} \int_{\theta=0}^{\pi/4} \int_{\rho=0}^{1} \sqrt{1-\rho^2\cos^2\theta}\, \rho\, d\rho\, d\theta$$

$$\begin{matrix} u = 1-\rho^2\cos^2\theta \\ du = -2\rho\cos^2\theta\, d\rho \end{matrix} \Rightarrow = \int_0^{\pi/4} \frac{1}{2\cos^2\theta} \left(\int_{\sin^2\theta}^{1} \sqrt{u}\, du \right) d\theta = \int_0^{\pi/4} \frac{1}{\cancel{2}\cos^2\theta} \left[\frac{\cancel{2} u^{3/2}}{3} \right]_{\sin^2\theta}^{1} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/4} \frac{1-\operatorname{sen}^3\theta}{\cos^2\theta}\, d\theta = \frac{1}{3}\int_0^{\pi/4} \frac{1}{\cos^2\theta}\, d\theta + \frac{1}{3}\int_0^{\pi/4} \frac{1-\cos^2\theta}{\cos^2\theta}\, d(\cos\theta)$$

$$= \frac{1}{3}\Big[\operatorname{tg}\theta\Big]_0^{\pi/4} + \frac{1}{3}\Big[-\frac{1}{\cos\theta} - \cos\theta\Big]_0^{\pi/4}$$

$$= \frac{1}{3} + \frac{1}{3}\left(-\sqrt{2} - \frac{\sqrt{2}}{2} + 2\right) = 1 - \frac{\sqrt{2}}{2}.$$

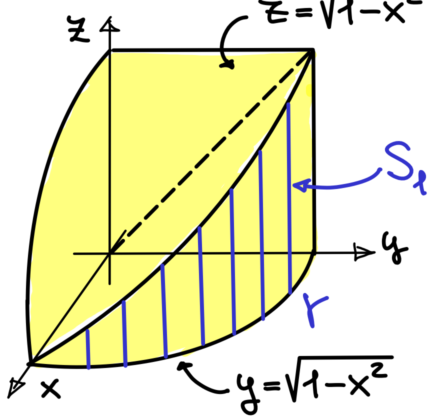
Infine

$$|D| = 16\,|D_1| = 8(2-\sqrt{2}).$$

2.e Calcolare l'area della superficie di

$$D = \{(x,y,z) : x^2+y^2 \le 1,\ x^2+z^2 \le 1\}.$$

La superficie S del solido D è composta da $8\cdot 2$ parti che, per le simmetrie rispetto ai piani $x=0, y=0, z=0, y=z$, hanno la stessa area. Una di queste parti è



$$S_1 = \{(x,y,z) : y = \sqrt{1-x^2},\ 0 \le z \le \sqrt{1-x^2},\ x \ge 0, y \ge 0\}.$$

Quindi

$$|S_1| = \int_\gamma f\, ds = \int_0^{\pi/2} f(\bar{\gamma}(t)) \cdot \|\bar{\gamma}'(t)\|\, dt$$

con $\bar{\gamma}(t) = (\cos t, \operatorname{sen} t)$ per $t \in [0, \frac{\pi}{2}]$ e $f(x,y) = \sqrt{1-x^2}$.

Così $\|\bar{\gamma}'(t)\|^2 = \|(-\operatorname{sen} t, \cos t)\|^2 = (-\operatorname{sen} t)^2 + (\cos t)^2 = 1$,

$$|S_1| = \int_0^{\pi/2} \underbrace{\sqrt{1-\cos^2 t}}_{|\operatorname{sen} t|} \cdot 1\, dt = \Big[-\cos t\Big]_0^{\frac{\pi}{2}} = 1 \quad \text{e} \quad |S| = 16\,|S_1| = 16.$$

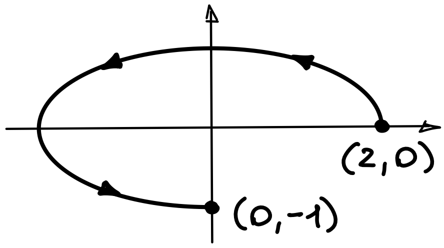
$\boxed{3.a}$ Calcolare $\int_\gamma \langle \vec{F}, d\vec{s}\rangle$ dove $\vec{F}(x,y)=(x+y,\,x-y)$

e $\vec{\gamma}(t)=(2\cos t, \sin t)$ per $t\in[0,\frac{3\pi}{2}]$.

$\vec{F}$ è $C^1(\mathbb{R}^2)$ ed è irrotazionale $\frac{\partial F_1}{\partial y}=1=\frac{\partial F_2}{\partial x}$.

Quindi $\vec{F}$ è conservativo in $\mathbb{R}^2$ (semplicemente connesso)

Potenziale: dobbiamo risolvere



$$\frac{\partial U}{\partial x}=x+y \quad e \quad \frac{\partial U}{\partial y}=x-y.$$

Allora

$$U(x,y)=\int (x+y)\,dx=\frac{x^2}{2}+xy+c(y) \quad e$$

$$\frac{\partial U}{\partial y}=\frac{\partial}{\partial y}\left(\frac{x^2}{2}+xy+c(y)\right)=0+x+c'(y)\stackrel{?}{=}x-y \Rightarrow c'(y)=-y$$

$$\Rightarrow c(y)=-\frac{y^2}{2}+c.$$

Per il calcolo poniamo $c=0$

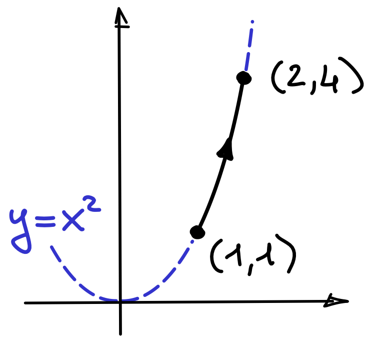
Così $U(x,y)=\frac{x^2}{2}+xy-\frac{y^2}{2}+\cancel{c}$ e

$$\int_\gamma \langle \vec{F}, d\vec{s}\rangle = U(\underbrace{\vec{\gamma}(\tfrac{3\pi}{2})}_{(0,-1)})-U(\underbrace{\vec{\gamma}(0)}_{(2,0)})=-\frac{1}{2}-2=-\frac{5}{2}.$$

$\boxed{3.b}$ Calcolare $\int_\gamma \langle \vec{F}, d\vec{s}\rangle$ dove $\vec{F}(x,y)=\left(\frac{1}{x+y},\,x\right)$

e $\vec{\gamma}(t)=(t,t^2)$ per $t\in[1,2]$.

Dato che $\vec{\gamma}'(t)=(1,2t)$, si ha



$$\int_\gamma \langle \vec{F}, d\vec{s}\rangle=\int_1^2 \left(\overset{\frac{1}{t}-\frac{1}{t+1}}{\frac{1}{t+t^2}}+2t\cdot t\right)dt$$

$$=\left[\log\left(\frac{t}{t+1}\right)+\frac{2t^3}{3}\right]_1^2$$

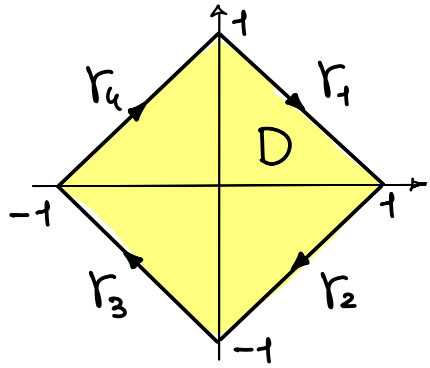
$$=\log\left(\frac{2}{3}\right)+\frac{16}{3}-\log\left(\frac{1}{2}\right)-\frac{2}{3}=\log\left(\frac{4}{3}\right)+\frac{14}{3}.$$

3.d Calcolare $\int_\gamma \langle \vec{F}, d\vec{s} \rangle$ dove $\vec{F}(x,y) = \left(\frac{x+y}{x+y+2}, -\frac{1}{x+y+2} \right)$

e γ è il bordo di

$$D = \{ (x,y) : |x| + |y| \leq 1 \}$$

percorso in senso orario.



Notiamo che in D $x+y+2 \neq 0$
e quindi $\vec{F}$ è continuo in D

Parametrizzazione di ∂D: per $t \in [0,1]$

$$\vec{\gamma}_1(t) = (t, 1-t) \qquad \vec{\gamma}_2(t) = (1-t, -t)$$

$$\vec{\gamma}_3(t) = (-t, -1+t) \qquad \vec{\gamma}_4(t) = (-1+t, t)$$

Ora si può fare il calcolo diretto con $\vec{F}$ oppure osservare che $\vec{F} = \vec{F}_1 + \vec{F}_2$,

$$\vec{F}_1(x,y) = \left(-\frac{1}{x+y+2}, 0 \right), \quad \vec{F}_2(x,y) = \left(1 - \frac{1}{x+y+2}, -\frac{1}{x+y+2} \right)$$

dove $\vec{F}_2$ è conservativo in D con potenziale
$U(x,y) = x - \log(|x+y+2|)$. Allora

$$\int_\gamma \langle \vec{F}, d\vec{s} \rangle = \int_\gamma \langle \vec{F}_1, d\vec{s} \rangle = \sum_{i=1}^{4} \int_{\gamma_i} \langle \vec{F}_1, d\vec{s} \rangle$$

$$= \int_0^1 \frac{-1}{3}(+1)\,dt + \int_0^1 \frac{-1}{3-2t}(-1)\,dt + \int_0^1 \frac{-1}{1}(-1)\,dt + \int_0^1 \frac{-1}{2t+1}(+1)\,dt$$

$$= -\frac{1}{3} + \left[-\frac{1}{2} \log|3-2t| \right]_0^1 + 1 - \left[\frac{1}{2} \log|2t+1| \right]_0^1$$

$$= \frac{2}{3} + \cancel{\frac{1}{2}\log 3} - \cancel{\frac{1}{2}\log 3} = \frac{2}{3}.$$