ANALISI MATEMATICA 2 - LEZIONE 21

ALCUNI ESERCIZI DEL FOGLIO 5

1.a $f(x,y,z) = x+z$ Max/min in $D = \left\{ (x,y,z) : \begin{array}{l} x^2+y^2+z^2=2 \\ z=x^2+y^2 \end{array} \right\}$

$f \in C^2(\mathbb{R}^3)$, D è chiuso e limitato. Poniamo

$$g_1(x,y,z) = x^2+y^2+z^2-2=0, \quad g_2(x,y,z)=x^2+y^2-z=0.$$

Regolarità:

$$\begin{bmatrix} g_{1x} & g_{1y} & g_{1z} \\ g_{2x} & g_{2y} & g_{2z} \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -1 \end{bmatrix}$$

Il rango non vale 2 se e solo se

$$\begin{cases} \det \begin{bmatrix} 2x & 2z \\ 2x & -1 \end{bmatrix} = 2x(-1-2z)=0 \\ \det \begin{bmatrix} 2y & 2z \\ 2y & -1 \end{bmatrix} = 2y(-1-2z)=0 \end{cases} \iff \begin{array}{l} (0,0,z) \notin D \\ (x,y,-\frac{1}{2}) \notin D \end{array}$$

Tutti i punti di D sono regolari.

Moltiplicatori:

$$\begin{cases} 1 = \lambda 2x + \mu 2x \\ 0 = \lambda 2y + \mu 2y \\ 1 = \lambda 2z - \mu \\ x^2+y^2+z^2=2 \\ z=x^2+y^2 \end{cases}$$

$\to 1 = 2x(\lambda+\mu) \to \lambda+\mu \neq 0$

$\to 2y\underbrace{(\lambda+\mu)}_{\neq 0}=0 \to y=0$

$$\begin{cases} y=0 \\ 1=2(\lambda+\mu)x \\ 1=2\lambda z-\mu \\ x^2+z^2=2 \\ z=x^2 \end{cases}$$

$x^2+z^2=2 \to z+z^2=2$

$z=1$: $x=1$, $x=-1$; $\quad z=-2=x^2$: $\emptyset$

Punti stazionari vincolati:

$(1,0,1)$ $\lambda=\frac{1}{2}, \mu=0$

$(-1,0,1)$ $\lambda=\frac{1}{6}, \mu=-\frac{2}{3}$

$\boxed{\begin{array}{l} f(1,0,1)=2 \\ f(-1,0,1)=0 \end{array}}$ $\begin{array}{l} \text{Max} \\ \text{Min} \end{array}$

$(1,0,1)$ punto di max assoluto

$(-1,0,1)$ punto di min assoluto

2.a

$$\iint_D \frac{e^{x/y}}{y^4}\,dx\,dy \qquad D=[0,2]\times[1,2]$$

$$=\int_{y=1}^{2}\frac{1}{y^4}\left(\int_{x=0}^{2} e^{x/y}\,dx\right)dy=\int_{y=1}^{2}\frac{1}{y^4}\left[y\,e^{x/y}\right]_0^2 dy=\int_1^2\frac{e^{2/y}-1}{y^3}\,dy$$

$t=\frac{1}{y}$ $\rightarrow$

$dt=-\frac{dy}{y^2}$

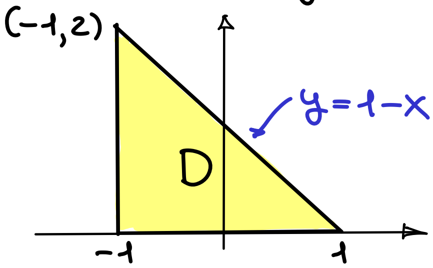
$$=\int_1^{1/2}\frac{e^{2t}-1}{1/t}(-dt)=\int_{1/2}^{1}\left(te^{2t}-t\right)dt$$

$$=\left[\frac{te^{2t}}{2}-\frac{e^{2t}}{4}-\frac{t^2}{2}\right]_{1/2}^{1}=\frac{e^2}{2}-\frac{e^2}{4}-\frac{1}{2}-\left(\cancel{\frac{e}{4}}-\cancel{\frac{e}{4}}-\frac{1}{8}\right)=\frac{e^2}{4}-\frac{3}{8}$$

dove $\int te^{2t}dt=\int t\,d\left(\frac{e^{2t}}{2}\right)=\frac{te^{2t}}{2}-\frac{e^{2t}}{4}+c.$

2.b

$$\iint_D (1-|x|-y)\,dx\,dy \qquad D=\left\{(x,y):\ \begin{matrix}x\geq -1,\ y\geq 0\\ x+y\leq 1\end{matrix}\right\}$$



$$=\int_{x=-1}^{1}\left(\int_{y=0}^{1-x}(1-|x|-y)\,dy\right)dx$$

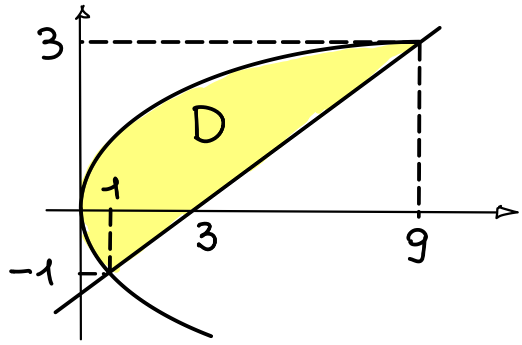
$$=\int_{-1}^{1}\left[(1-|x|)y-\frac{y^2}{2}\right]_0^{1-x}dx=\int_{-1}^{1}\left((1-|x|)(1-x)-\frac{1}{2}(1-x)^2\right)dx$$

$\rightarrow \underset{p}{1}-\underset{p}{|x|}-\cancel{x}+\underset{d}{x|x|}-\underset{p}{\frac{1}{2}}+\cancel{x}-\underset{p}{\frac{x^2}{2}}$

$$=2\int_0^1\left(\frac{1}{2}-x-\frac{x^2}{2}\right)dx=\left[x-x^2-\frac{x^3}{3}\right]_0^1=-\frac{1}{3}.$$

2.c

$$\iint_D \frac{1}{(2y-x+8)^2}\,dx\,dy \qquad D=\{(x,y): y^2 \le x \le 2y+3\}$$



$$= \int_{y=-1}^{3} \left(\int_{x=y^2}^{2y+3} \frac{1}{(2y-x+8)^2}\,dx \right) dy$$

$$= \int_{y=-1}^{3} \left[\frac{1}{2y-x+8} \right]_{y^2}^{2y+3} dy = \int_{-1}^{3} \left(\frac{1}{5} - \frac{1}{\underbrace{2y-y^2+8}_{-(y+2)(y-4)}} \right) dy$$

$$= \frac{3-(-1)}{5} + \frac{1}{6}\int_{-1}^{3} \left(\frac{1}{y-4} - \frac{1}{y+2} \right) dy = \frac{4}{5} + \frac{1}{6}\left[\log\left| \frac{y-4}{y+2} \right| \right]_{-1}^{3}$$

$$= \frac{4}{5} + \frac{1}{6}\left(\log\left(\frac{1}{5}\right) - \log(5) \right) = \frac{4}{5} - \frac{\log(5)}{3}$$

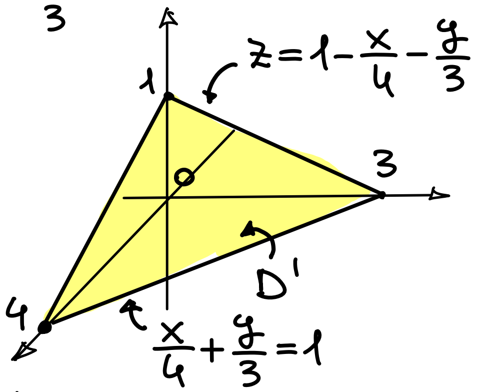
$\boxed{3.a}$ $D=\left\{(x,y,z) : \frac{|x|}{4}+\frac{|y|}{3}+|z|\leq 1\right\}$ Volume?

D è simmetrico rispetto ai piani $x=0, y=0, z=0$ pertanto $|D|=8|E|$ dove

$$E=\left\{(x,y,z) : \frac{x}{4}+\frac{y}{3}+z\leq 1,\ x,y,z\geq 0\right\}$$

1) Per fili con $\varphi_1(x,y)=0$, $\varphi_2(x,y)=1-\frac{x}{4}-\frac{y}{3}$

e $D'=\left\{(x,y): \frac{x}{4}+\frac{y}{3}\leq 1, x,y\geq 0\right\}$.



$$|D|=8|E|=8\iint_{D'}\left(1-\frac{x}{4}-\frac{y}{3}\right)dx\,dy$$

$$=8\int_{x=0}^{4}\left(\int_{y=0}^{3-\frac{3x}{4}}\left(1-\frac{x}{4}-\frac{y}{3}\right)dy\right)dx = 8\int_{u=0}^{1}\left(\int_{v=0}^{1-u}(1-u-v)3\,dv\right)4\,du$$

$\begin{cases}x=4u\\ y=3v\end{cases}$

$$=8\cdot 12\int_0^1\left[v-uv-\frac{v^2}{2}\right]_0^{1-u}du = 8\cdot 12\int_0^1\left((1-u)^2-\frac{(1-u)^2}{2}\right)du$$

$$=\frac{8\cdot 12}{2}\int_0^1(1-u)^2du = 48\left[-\frac{(1-u)^3}{3}\right]_0^1 = \frac{48}{3}=16.$$

2) Per sezioni con $z\in[0,1]$ e $S_z=\left\{(x,y): \frac{x}{4}+\frac{y}{3}\leq 1-z\right\}$

triangolo rettangolo con cateti $4(1-z)$ e $3(1-z)$

$$|D|=8|E|=8\int_{z=0}^{1}\left(\iint_{S_z}1\,dx\,dy\right)dz$$

$$=8\int_0^1|S_z|\,dz=8\int_0^1\frac{4\cdot 3(1-z)^2}{2}dz = 8\cdot 6\left[-\frac{(1-z)^3}{3}\right]_0^1$$

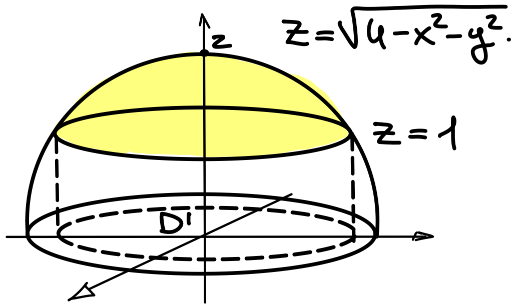
$$=8\,\cancel{6}^{\,2}\,\frac{1}{\cancel{3}}=16.$$

$\boxed{3.b}$ $D=\{(x,y,z): x^2+y^2+z^2\leq 4,\ z\geq 1\}$ Volume?

calotta

1) Per fili con $D'=\{x^2+y^2+1\leq 4\}$

$\varphi_1(x,y)=1$ e $\varphi_2(x,y)=\sqrt{4-x^2-y^2}$.



$$|D|=\iint_{D'}\left(\sqrt{4-x^2-y^2}-1\right)dx\,dy$$

$$\overset{CP}{=}\int_{\rho=0}^{\sqrt{3}}\int_{\theta=0}^{2\pi}\left(\sqrt{4-\rho^2}-1\right)\rho\,d\rho\,d\theta=2\pi\int_0^{\sqrt{3}}\left(\sqrt{4-\rho^2}-1\right)\rho\,d\rho$$

$t=4-\rho^2$
$dt=-2\rho\,d\rho$

$$=\cancel{2}\pi\int_4^1\left(t^{1/2}-1\right)\left(-\frac{dt}{\cancel{2}}\right)=\pi\left[\frac{2t^{3/2}}{3}-t\right]_1^4$$

$$=\pi\left(\left(\frac{16}{3}-4\right)-\left(\frac{2}{3}-1\right)\right)=\frac{5}{3}\pi.$$

2) Per sezioni con $z\in[1,2]$ e $S_z=\{(x,y): x^2+y^2\leq 4-z^2\}$

cerchio di raggio $\sqrt{4-z^2}$

$$|D|=\int_{z=1}^{2}\left(\iint_{S_z}1\,dx\,dy\right)dz=\int_1^2|S_z|\,dz=$$

$$=\int_1^2\pi(4-z^2)\,dz=\pi\left[4z-\frac{z^3}{3}\right]_1^2$$

$$=\pi\left(8-\frac{8}{3}-4+\frac{1}{3}\right)=\frac{5}{3}\pi.$$

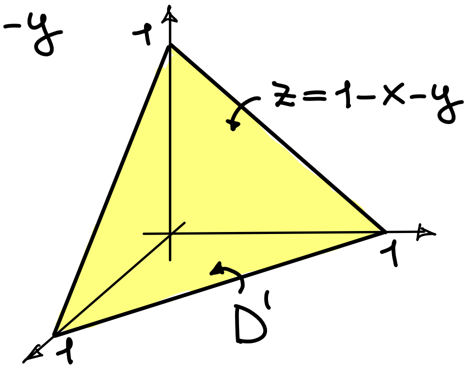
$\boxed{4.a}$

$$\iiint_D \frac{1}{(x+y+z+1)^3}\,dx\,dy\,dz$$

dove $D=\{(x,y,z): x,y,z\geq 0,\ x+y+z\leq 1\}$.

Per fili con $\varphi_1(x,y)=0$, $\varphi_2(x,y)=1-x-y$

e $D'=\{(x,y): x,y\geq 0,\ x+y\leq 1\}$



$$=\iint_{D'}\left(\int_{z=0}^{1-x-y}\frac{1}{(x+y+z+1)^3}\,dz\right)dx\,dy$$

$$=\iint_{D'}\left[\frac{-1/2}{(x+y+z+1)^2}\right]_0^{1-x-y}dx\,dy=\frac{1}{2}\iint_{D'}\left(\frac{1}{(x+y+1)^2}-\frac{1}{4}\right)dx\,dy$$

$$=\frac{1}{2}\int_{x=0}^{1}\left(\int_{y=0}^{1-x}\frac{1}{(x+y+1)^2}\,dy\right)dx-\frac{|D'|}{8}\qquad |D'|=\frac{1}{2}$$

$$=\frac{1}{2}\int_0^1\left[-\frac{1}{x-y+1}\right]_0^{1-x}dx-\frac{1}{16}=\frac{1}{2}\int_0^1\left(\frac{1}{x+1}-\frac{1}{2}\right)dx-\frac{1}{16}$$

$$=\frac{1}{2}\Big[\log(x+1)\Big]_0^1-\frac{1}{4}-\frac{1}{16}=\frac{\log(2)}{2}-\frac{5}{16}.$$