ANALISI MATEMATICA 2 - LEZIONE 20

INTEGRALI TRIPLI

Le definizioni e le proprietà viste per gli integrali doppi si possono estendere agli integrali tripli

$$\iiint_D f(x,y,z)\,dx\,dy\,dz$$

con $D \subseteq \mathbb{R}^3$ e $f: D \to \mathbb{R}$.

Si noti che $|D| = \iiint_D 1\,dx\,dy\,dz$ indica il volume di D.

Vediamo i risultati principali utili per il calcolo.

TEOREMA (FORMULE DI RIDUZIONE)

Sia f continua in $D \subseteq \mathbb{R}^3$.

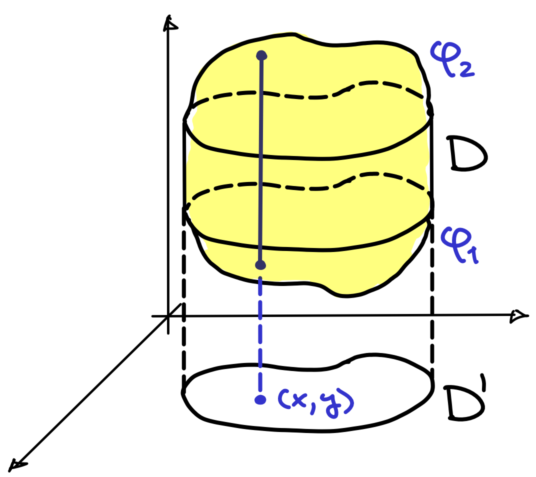
1) INTEGRAZIONE PER FILI

Se $\exists$ ins. misurabile $D' \subseteq \mathbb{R}^2$ e $\exists\, \varphi_1, \varphi_2 \in C(D')$ tali che

$$D = \{(x,y,z) : (x,y) \in D',\ z \in [\varphi_1(x,y), \varphi_2(x,y)]\}$$

allora

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iint_{D'} \left(\int_{z=\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\,dz \right) dx\,dy$$



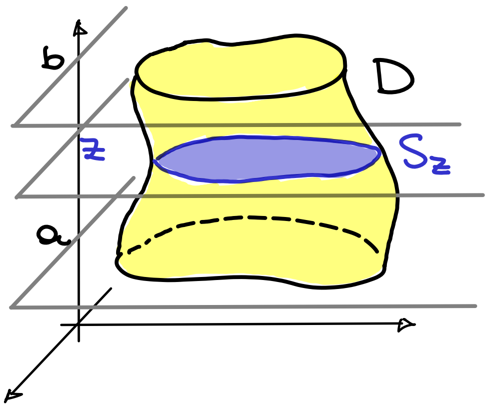
2) INTEGRAZIONE PER SEZIONI

Se $\forall z \in [a,b]$ $\exists$ ins. misurabile $S_z \subseteq \mathbb{R}^2$ tale che

$$D = \{(x,y,z) : z \in [a,b] \text{ e } (x,y) \in S_z\}$$

allora

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \int_{z=a}^{b} \left(\iint_{S_z} f(x,y,z)\,dx\,dy \right) dz$$



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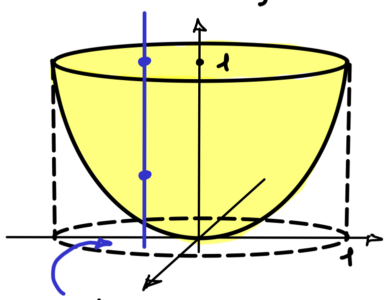
- $\displaystyle\iiint_D |x| z\,dx\,dy\,dz \qquad D = \{(x,y,z) : x^2+y^2 \le z \le 1\}.$

1) Integrazione per fili



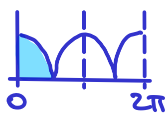
$$D' = \{(x,y) : x^2+y^2 \le 1\}$$

$$= \iint_{D'} |x| \left(\int_{z=x^2+y^2}^{1} z\,dz \right) dx\,dy$$

$$= \iint_{D'} |x| \cdot \left[\frac{z^2}{2} \right]_{x^2+y^2}^{1} dx\,dy = \frac{1}{2} \iint_{D'} |x| \cdot \left(1 - (x^2+y^2)^2\right) dx\,dy$$

$$\overset{CP}{=} \frac{1}{2} \int_{\rho=0}^{1} \int_{\theta=0}^{2\pi} \rho |\cos\theta| \cdot (1-\rho^4)\,\rho\,d\rho\,d\theta$$

$$\int_0^{2\pi} |\cos\theta|\,d\theta$$



$$= 4\int_0^{\pi/2} \cos\theta\,d\theta = 4\left[\sin\theta\right]_0^{\pi/2} = 4$$

$$= \frac{\cancel{4}^{2}}{\cancel{2}_{1}} \int_0^1 \rho^2 (1-\rho^4)\,d\rho = 2\left[\frac{\rho^3}{3} - \frac{\rho^7}{7} \right]_0^1 = \frac{2(7-3)}{21} = \frac{8}{21}.$$

2) Integrazione per sezioni

$$= \int_{z=0}^{1} z \left(\iint_{S_z} |x|\,dx\,dy \right) dz$$

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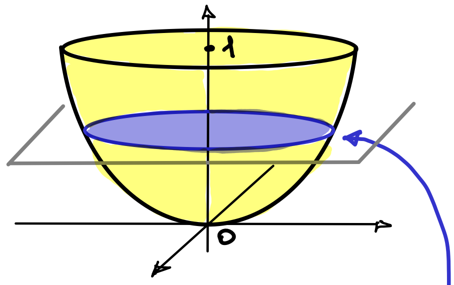


$S_z = \{(x,y) : x^2+y^2 \leq z\}$

$$= \int_{z=0}^{1} z \left(\int_{\rho=0}^{\sqrt{z}} \int_{\theta=0}^{2\pi} \rho|\cos\theta| \cdot \rho \, d\rho\, d\theta \right) dz$$

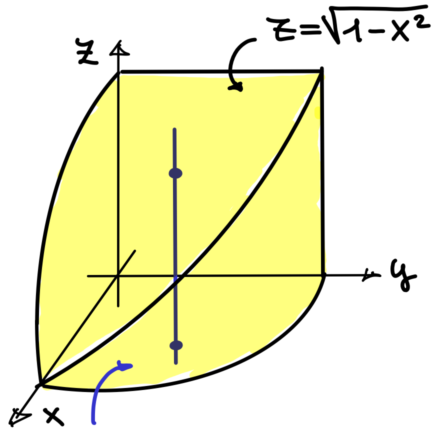
$$= \int_{z=0}^{1} z \left(\int_{\rho=0}^{\sqrt{z}} 4\rho^2 d\rho \right) dz = 4 \int_{z=0}^{1} z \left[\frac{\rho^3}{3} \right]_0^{\sqrt{z}} dz = \frac{4}{3} \int_0^1 z^{1+3/2} dz$$

$$= \frac{4}{3} \left[\frac{2}{7} z^{\frac{7}{2}} \right]_0^1 = \frac{8}{21}.$$

- Calcolare il volume dell'insieme

$$D = \{ (x,y,z) : x^2+y^2 \leq 1,\ x^2+z^2 \leq 1 \}.$$

D è l'intersezione di due cilindri



Sia $E = D \cap \{(x,y,z) : x,y,z \geq 0\}$

Per simmetria $|D| = 8|E|$.

1) Integrazione per fili

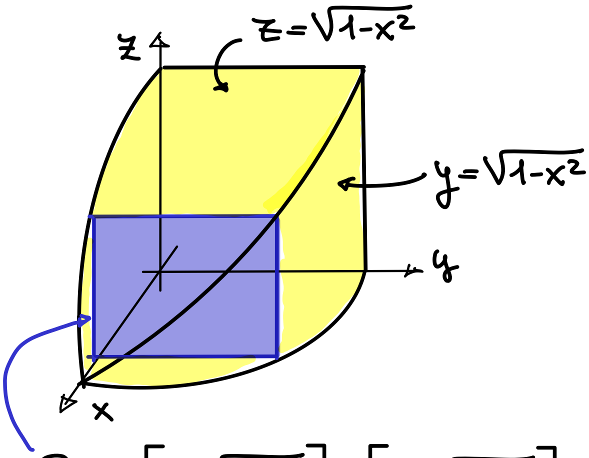
$D' = \{(x,y) : x^2+y^2 \leq 1,\ x,y \geq 0\}$

$$|D| = 8|E| = 8 \iint_{D'} \left(\int_{z=0}^{\sqrt{1-x^2}} 1\, dz \right) dx\,dy$$

$$= 8 \iint_{D'} \left(\sqrt{1-x^2} \right) dx\,dy = 8 \int_{x=0}^{1} \sqrt{1-x^2} \left(\int_{y=0}^{\sqrt{1-x^2}} 1\, dy \right) dx$$

$$= 8 \int_0^1 (1-x^2)\, dx = 8 \left[x - \frac{x^3}{3} \right]_0^1 = \frac{16}{3}.$$

2) Integrazione per sezioni lungo l'asse x



$$S_x = [0,\sqrt{1-x^2}]\times[0,\sqrt{1-x^2}]$$

$$|D| = 8\int_{x=0}^{1}\left(\iint_{S_x} 1\,dy\,dz\right)dx$$

$$= 8\int_{x=0}^{1} |S_x|\,dx$$

$$= 8\int_0^1 \left(\sqrt{1-x^2}\right)^2 dx = 8\left[x-\frac{x^3}{3}\right]_0^1 = \frac{16}{3}.$$

$\boxed{\text{TEOREMA}}$ (CAMBIO DI VARIABILI PER INTEGRALI TRIPLI)

Sia $\vec{\Phi}: A \to \vec{\Phi}(A)$ una funzione biunivoca con A aperto di $\mathbb{R}^3$ e $\vec{\Phi} \in C^1(A)$. Sia $D \subseteq \vec{\Phi}(A)$ un insieme limitato e misurabile e sia f una funzione continua e limitata in D. Se $\det(J_{\vec{\Phi}}(u,v,w)) \neq 0$ in $\vec{\Phi}^{-1}(D)\setminus E$ con $|E|=0$ allora

$$\begin{cases} x = \varphi_1(u,v,w) \\ y = \varphi_2(u,v,w) \\ z = \varphi_3(u,v,w) \end{cases} \quad \vec{\Phi}(u,v,w)$$

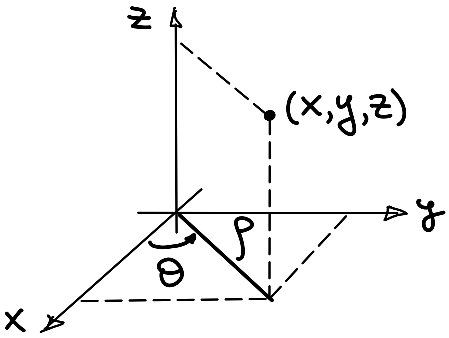
$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{\vec{\Phi}^{-1}(D)} f(\vec{\Phi}(u,v,w))\left|\det(J_{\vec{\Phi}}(u,v,w))\right| du\,dv\,dw$$

dove la MATRICE JACOBIANA di $\vec{\Phi}$ è

$$J_{\vec{\Phi}}(u,v,w) = \begin{bmatrix} \frac{\partial \varphi_1}{\partial u}(u,v,w) & \frac{\partial \varphi_1}{\partial v}(u,v,w) & \frac{\partial \varphi_1}{\partial w}(u,v,w) \\ \frac{\partial \varphi_2}{\partial u}(u,v,w) & \frac{\partial \varphi_2}{\partial v}(u,v,w) & \frac{\partial \varphi_2}{\partial w}(u,v,w) \\ \frac{\partial \varphi_3}{\partial u}(u,v,w) & \frac{\partial \varphi_3}{\partial v}(u,v,w) & \frac{\partial \varphi_3}{\partial w}(u,v,w) \end{bmatrix}$$

Vediamo il calcolo di $\det(J_\Phi)$ per due cambi di coordinate in dimensione tre.

1) COORDINATE CILINDRICHE (ρ,θ,z):



$$\begin{cases} x=\varphi_1(\rho,\theta,z)=\rho\cos\theta \\ y=\varphi_2(\rho,\theta,z)=\rho\,\mathrm{sen}\,\theta \\ z=\varphi_3(\rho,\theta,z)=z \end{cases}$$

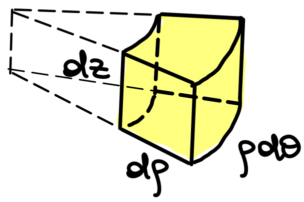
$\rho=\sqrt{x^2+y^2}$

$\theta\in[0,2\pi)$

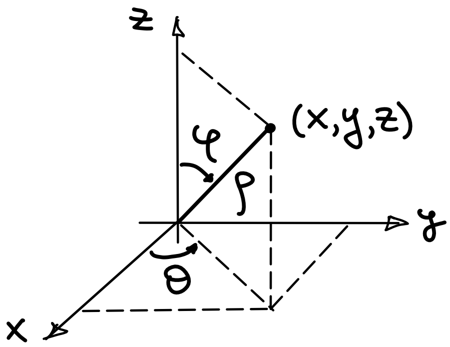
$$J_\Phi(\rho,\theta,z)=\begin{bmatrix} \cos\theta & -\rho\,\mathrm{sen}\,\theta & 0 \\ \mathrm{sen}\,\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

e dunque $|\det(J_\Phi(\rho,\theta,z))|=\rho$.

L'elemento infinitesimo di volume è $\Rightarrow \rho\,d\rho\,d\theta\,dz$



2) COORDINATE SFERICHE (ρ,θ,φ):



$$\begin{cases} x=\varphi_1(\rho,\theta,\varphi)=\rho\,\mathrm{sen}\,\varphi\cos\theta \\ y=\varphi_2(\rho,\theta,\varphi)=\rho\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta \\ z=\varphi_3(\rho,\theta,\varphi)=\rho\cos\varphi \end{cases}$$

$\rho=\sqrt{x^2+y^2+z^2}$

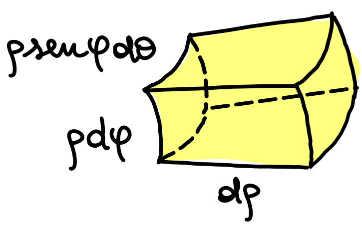
$\theta\in[0,2\pi)$

$\varphi\in[0,\pi]$

$$J_\Phi(\rho,\theta,\varphi)=\begin{bmatrix} \mathrm{sen}\,\varphi\cos\theta & -\rho\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta & \rho\cos\varphi\cos\theta \\ \mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta & \rho\,\mathrm{sen}\,\varphi\cos\theta & \rho\cos\varphi\,\mathrm{sen}\,\theta \\ \cos\varphi & 0 & -\rho\,\mathrm{sen}\,\varphi \end{bmatrix}$$

e dunque $|\det(J_\Phi(\rho,\theta,\varphi))|=\rho^2\,\mathrm{sen}\,\varphi$.

L'elemento infinitesimo di volume è $\Rightarrow \rho^2\,\mathrm{sen}\,\varphi\,d\rho\,d\theta\,d\varphi$



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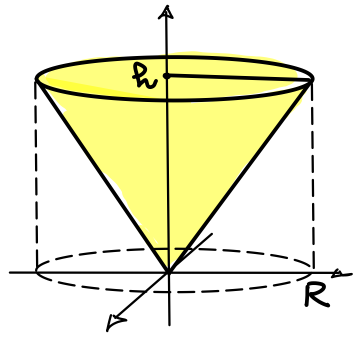

- Calcolare il volume dell'insieme

$$D=\left\{(x,y,z): \frac{x^2+y^2}{R^2}\leq\frac{z^2}{h^2},\ 0\leq z\leq h\right\}$$

doppio cono infinito

$\vec{\Phi}$ coordinate cilindriche $\quad \vec{\Phi}^{-1}(D)=\left\{(\rho,\theta,z): \begin{array}{l}\rho\in[0,R]\\ \theta\in[0,2\pi)\end{array},\ \frac{h}{R}\rho\leq z\leq h\right\}$

$$|D|=\iiint_D 1\,dx\,dy\,dz=\iiint_{\vec{\Phi}^{-1}(D)} 1\,\rho\,d\rho\,d\theta\,dz$$

$$=\iint_{[0,R]\times[0,2\pi)}\left(\int_{z=\frac{h}{R}\rho}^{h} dz\right)\rho\,d\rho\,d\theta=\int_{\rho=0}^{R}\left(\int_{\theta=0}^{2\pi} d\theta\right)\left(h-\frac{h}{R}\rho\right)\rho\,d\rho$$

$$=2\pi h\int_0^R\left(\rho-\frac{1}{R}\rho^2\right)d\rho=2\pi h\left[\frac{\rho^2}{2}-\frac{1}{R}\cdot\frac{\rho^3}{3}\right]_0^R=\cancel{2}\pi h\frac{R^2}{\cancel{6}\,3}=\frac{\pi R^2 h}{3}.$$

- $\displaystyle\iiint_D z\,dx\,dy\,dz \qquad D=\left\{(x,y,z): \begin{array}{l}x^2+y^2+z^2\leq R^2\\ z\geq 0\end{array}\right\}$

semisfera

$\vec{\Phi}$ coordinate sferiche $\quad \vec{\Phi}^{-1}(D)=\left\{(\rho,\theta,\varphi)\in[0,R]\times[0,2\pi)\times\left[0,\frac{\pi}{2}\right]\right\}$

$$\iiint_D z\,dx\,dy\,dz=\iiint_{\vec{\Phi}^{-1}(D)}\rho\cos\varphi\cdot\rho^2\operatorname{sen}\varphi\,d\rho\,d\theta\,d\varphi$$

$$=\int_0^R\rho^3\,d\rho\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\underbrace{\cos\varphi\operatorname{sen}\varphi\,d\varphi}_{\operatorname{sen}\varphi\, d(\operatorname{sen}\varphi)}=2\pi\left[\frac{\rho^4}{4}\right]_0^R\cdot\left[\frac{\operatorname{sen}^2\varphi}{2}\right]_0^{\frac{\pi}{2}}$$

$$=\cancel{2}\pi\cdot\frac{R^4}{4}\cdot\frac{1}{\cancel{2}}=\frac{\pi}{4}R^4.$$