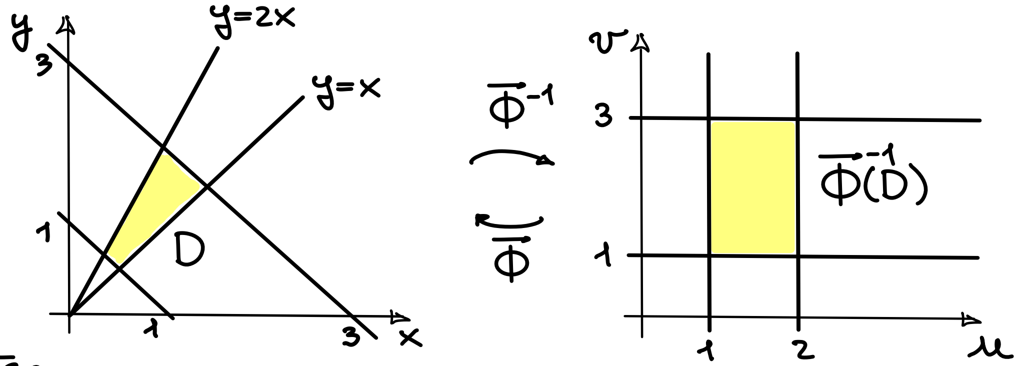
ANALISI MATEMATICA 2 - LEZIONE 19

ESEMPI

- $\displaystyle\iint_D \frac{1}{xy}\,dx\,dy \qquad D=\{(x,y): 0\le x\le y\le 2x,\ 1\le x+y\le 3\}.$

Poniamo

$$\begin{cases} u=\dfrac{y}{x} \\ v=x+y \end{cases} \quad \vec{\Phi}^{-1}(x,y) \quad \text{da cui} \quad \begin{cases} x=\dfrac{v}{u+1} \\ y=\dfrac{uv}{u+1} \end{cases} \quad \vec{\Phi}(u,v)$$



Inoltre

$$\det(J_{\vec{\Phi}}(u,v))=\det\left(\begin{bmatrix} -\dfrac{v}{(u+1)^2} & \dfrac{1}{u+1} \\ \dfrac{v}{(u+1)^2} & \dfrac{u}{u+1}\end{bmatrix}\right)=\frac{-vu-v}{(u+1)^3}=\frac{-v}{(u+1)^2}$$

Allora

$$\iint_D \frac{1}{xy}\,dx\,dy=\iint_{\vec{\Phi}^{-1}(D)} \frac{u+1}{v}\cdot\frac{u+1}{uv}\cdot\left|\frac{-v}{(u+1)^2}\right| du\,dv$$

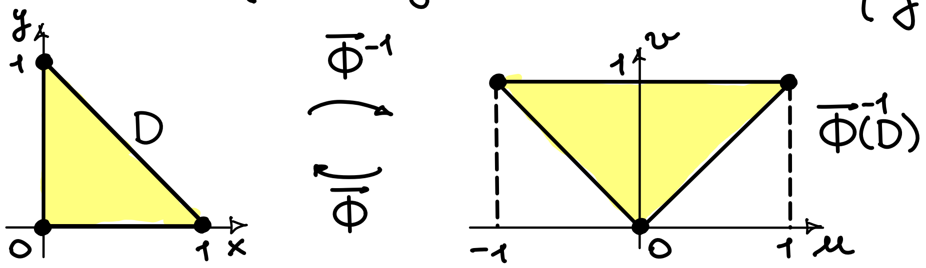
$$=\int_{u=1}^{2}\left(\int_{v=1}^{3}\frac{\cancel{(u+1)^2}}{u\,v^{\cancel{2}}}\cdot\frac{\cancel{v}}{\cancel{(u+1)^2}}\,dv\right)du$$

$$=\int_1^2 \frac{1}{u}\Big[\log(v)\Big]_1^3 du$$

$$=\log(3)\cdot\Big[\log(u)\Big]_1^2=\log(3)\cdot\log(2).$$

- $$\iint_D \frac{(x-y)^2}{1+x+y}\,dx\,dy \qquad D=\{(x,y): x\geq 0,\ y\geq 0,\ x+y\leq 1\}.$$

Poniamo $\begin{cases} u = x-y \\ v = x+y \end{cases}$ $\vec{\Phi}^{-1}(x,y)$ da cui $\begin{cases} x = \dfrac{u+v}{2} \\ y = \dfrac{v-u}{2} \end{cases}$ $\vec{\Phi}(u,v)$



$(0,0) \xrightarrow{\vec{\Phi}} (0,0)$

$(1,1) \rightarrow (1,0)$

$(-1,1) \rightarrow (0,1)$

$\vec{\Phi}$ è lineare

Inoltre

$$\det(J_{\vec{\Phi}}(u,v)) = \det\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}.$$

Allora

$$\iint_D \frac{(x-y)^2}{1+x+y}\,dx\,dy = \iint_{\vec{\Phi}^{-1}(D)} \frac{u^2}{1+v}\left|\frac{1}{2}\right| du\,dv$$

u-semplice

$$= \frac{1}{2}\int_{v=0}^{1}\left(\int_{u=-v}^{v} \frac{u^2}{1+v}\,du\right)dv$$

$$= \frac{1}{2}\int_{v=0}^{1} \frac{1}{1+v}\left[\frac{u^3}{3}\right]_{-v}^{v} dv$$

$$= \frac{1}{3}\int_0^1 \frac{v^3}{1+v}\,dv$$

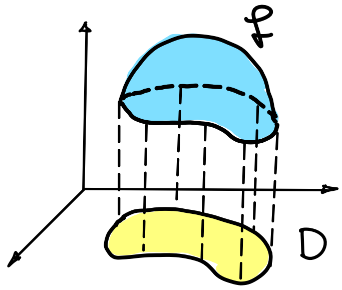
$$= \frac{1}{3}\int_0^1 \left(1 - v + v^2 - \frac{1}{1+v}\right) dv$$

$$= \frac{1}{3}\left[v - \frac{v^2}{2} + \frac{v^3}{3} - \log(1+v)\right]_0^1$$

$$= \frac{1}{3}\left(1-\frac{1}{2}+\frac{1}{3}-\log(2)\right) = \frac{5}{18} - \frac{\log(2)}{3}.$$

OSSERVAZIONE

Se $f(x,y) \geq 0$ per ogni $(x,y) \in D$ allora

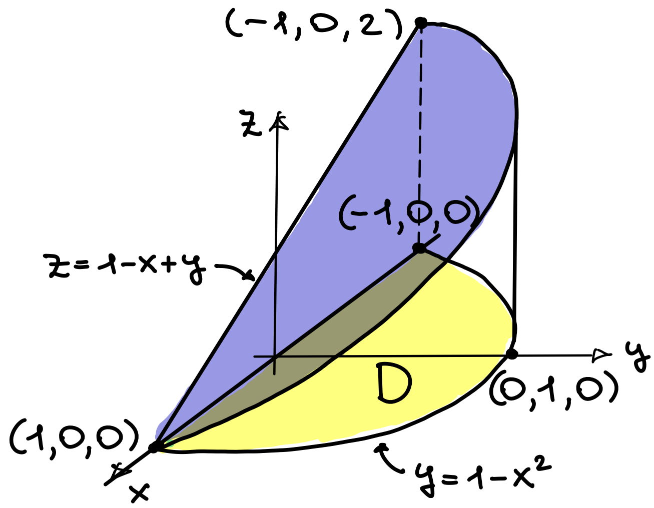


$$\iint_D f(x,y)\,dx\,dy = \text{Volume}(E) = |E|$$

dove $E = \{(x,y,z) : (x,y) \in D,\ 0 \leq z \leq f(x,y)\}$.

- Calcolare il volume di

$$E = \{(x,y,z) : y \geq 0,\ z \geq 0,\ x - y + z \leq 1,\ y \leq 1 - x^2\}$$



Consideriamo

$$D = \{(x,y) : 0 \leq y \leq 1 - x^2\}$$

e

$$z = f(x,y) = 1 - x + y.$$

Si noti che se $(x,y) \in D$ allora

$$f(x,y) \underset{y \geq 0}{\geq} 1 - x \underset{x \leq 1}{\geq} 0.$$

$$|E| = \iint_D (1 - x + y)\,dx\,dy = \int_{x=-1}^{1} \left(\int_{y=0}^{1-x^2} (1 - x + y)\,dy \right) dx$$

[−1,1] è simmetrico

$$= \int_{-1}^{1} \left[y - xy + \frac{y^2}{2} \right]_0^{1-x^2} dx = \int_{-1}^{1} \Big(\underbrace{(1-x^2)}_{\text{pari}} \underbrace{- x(1-x^2)}_{\text{dispari}} + \underbrace{\frac{(1-x^2)^2}{2}}_{\text{pari}} \Big) dx$$

$$= \int_0^1 \left(2(1-x^2) + (1 - 2x^2 + x^4) \right) dx = \int_0^1 (3 - 4x^2 + x^4)\,dx$$

$$= \left[3x - \frac{4x^3}{3} + \frac{x^5}{5} \right]_0^1 = \frac{45 - 20 + 3}{15} = \frac{28}{15}.$$

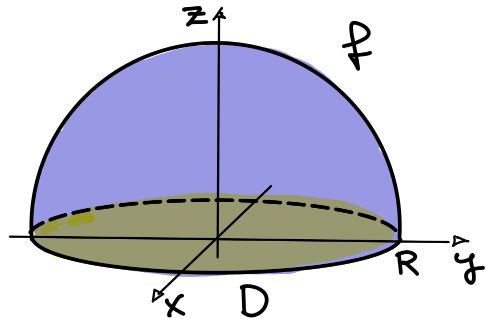
- Calcolare il volume di una sfera di raggio R.

La sfera di raggio R e centro (0,0,0) ha equazione

$$x^2+y^2+z^2=R^2.$$

Sia $E=\{(x,y,z): x^2+y^2+z^2\leq R^2\}$ e consideriamo la semisfera data dal grafico della funzione

$$z=f(x,y)=\sqrt{R^2-x^2-y^2} \quad \text{in} \quad D=\{(x,y): x^2+y^2\leq R^2\}.$$



Allora

$$|E|=2\iint_D f(x,y)\,dx\,dy$$

$$=2\iint_{\{x^2+y^2\leq R^2\}}\sqrt{R^2-x^2-y^2}\,dx\,dy$$

$$\overset{CP}{=}2\iint_{[0,R]\times[0,2\pi]}\sqrt{R^2-\rho^2}\,\rho\,d\rho\,d\theta$$

$$=2\int_{\rho=0}^{R}\sqrt{R^2-\rho^2}\left(\int_{\theta=0}^{2\pi}d\theta\right)\rho\,d\rho$$

$$=4\pi\int_0^R\sqrt{R^2-\rho^2}\,\rho\,d\rho$$

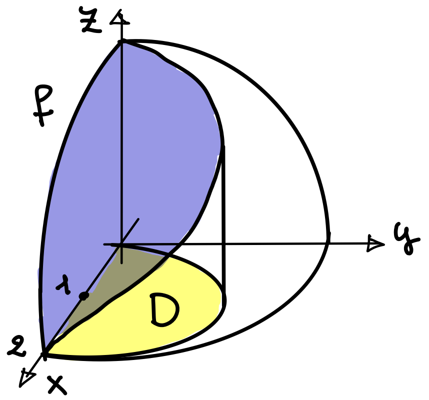
$t=R^2-\rho^2$
$dt=-2\rho\,d\rho$

$$=\cancel{4}^{\,2}\pi\int_{R^2}^{0}\sqrt{t}\left(\frac{dt}{-\cancel{2}}\right)=2\pi\left[\frac{2t^{3/2}}{3}\right]_0^{R^2}=\frac{4}{3}\pi R^3.$$

- Calcolare il volume dell'insieme

$$E=\{(x,y,z):x,y,z\geq 0,\ \underbrace{x^2+y^2+z^2\leq 4}_{\text{sfera}},\ x^2+y^2\leq 2x\}$$

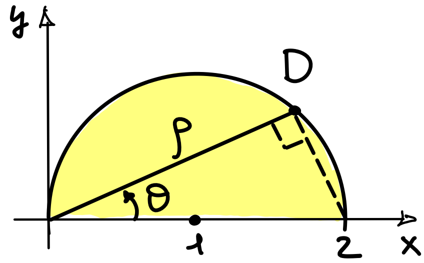
$\hookrightarrow (x-1)^2+y^2\leq 1$ cilindro



Consideriamo la funzione

$$z=f(x,y)=\sqrt{4-x^2-y^2}$$

in $D=\{(x,y): x^2+y^2\leq 2x,\ y\geq 0\}$.



In coordinate polari

$$\vec{\Phi}^{-1}(D)=\left\{(\rho,\theta):\ \begin{array}{l}0\leq\rho\leq 2\cos\theta\\ \theta\in[0,\frac{\pi}{2}]\end{array}\right\}$$

Allora

$$|E|=\iint_D\sqrt{4-x^2-y^2}\,dx\,dy\overset{CP}{=}\iint_{\vec{\Phi}^{-1}(D)}\sqrt{4-\rho^2}\cdot\rho\,d\rho\,d\theta$$

$$=\int_{\theta=0}^{\pi/2}\left(\int_{\rho=0}^{2\cos\theta}\sqrt{4-\rho^2}\cdot\rho\,d\rho\right)d\theta$$

$t=4-\rho^2$
$dt=-2\rho\,d\rho$

$$=\int_0^{\pi/2}\left(\int_{t=4}^{4\sin^2\theta}\sqrt{t}\left(-\frac{dt}{2}\right)\right)d\theta=\int_0^{\pi/2}\left[\frac{t^{3/2}}{3}\right]_{4\sin^2\theta}^{4}d\theta$$

$$=\frac{8}{3}\int_0^{\pi/2}(1-\sin^3\theta)\,d\theta$$

$$=\frac{8}{3}\cdot\frac{\pi}{2}+\frac{8}{3}\int_0^{\pi/2}(1-\cos^2\theta)\,d(\cos\theta)$$

$$=\frac{4\pi}{3}+\frac{8}{3}\left[\cos\theta-\frac{\cos^3\theta}{3}\right]_0^{\pi/2}=\frac{4\pi}{3}-\frac{16}{9}.$$

- Calcolare

1) $\displaystyle\lim_{n\to\infty}\iint_{D_n} e^{-(x^2+y^2)}\,dx\,dy$ con $D_n=\{(x,y): x^2+y^2\le n^2\}$.

2) $\displaystyle\lim_{n\to\infty}\iint_{Q_n} e^{-(x^2+y^2)}\,dx\,dy$ con $Q_n=[-n,n]\times[-n,n]$.

1) $$\iint_{D_n} e^{-(x^2+y^2)}\,dx\,dy \overset{CP}{=} \int_{\rho=0}^{n} e^{-\rho^2}\left(\int_{\theta=0}^{2\pi} d\theta\right)\rho\,d\rho$$
$$=2\pi\int_0^n e^{-\rho^2}\rho\,d\rho = \cancel{2}\pi\left[-\frac{e^{-\rho^2}}{\cancel{2}}\right]_0^n=\pi(1-e^{-n^2})\xrightarrow{n\to\infty}\pi.$$

non dipende da x

2) $$\iint_{Q_n} e^{-(x^2+y^2)}\,dx\,dy = \int_{x=-n}^{n} e^{-x^2}\cdot\left(\int_{y=-n}^{n} e^{-y^2}\,dy\right)dx$$
$$=\left(\int_{-n}^{n} e^{-x^2}dx\right)\cdot\left(\int_{-n}^{n} e^{-y^2}dy\right)\xrightarrow{n\to\infty}\left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2.$$

OSSERVAZIONE

I due limiti ottenuti in 1) e 2) sono uguali? SÌ

$$0\le\iint_{Q_n} e^{-(x^2+y^2)}\,dx\,dy-\iint_{D_n} e^{-(x^2+y^2)}\,dx\,dy=\iint_{Q_n\setminus D_n} e^{-(x^2+y^2)}\,dx\,dy$$

$Q_n\supseteq D_n$
e $f\ge 0$

∂D_n

$$\le \sup_{Q_n\setminus D_n} e^{-(x^2+y^2)}\cdot|Q_n\setminus D_n| = e^{-n^2}\cdot(4n^2-\pi n^2)$$
$$\to 0 \text{ per } n\to\infty$$

Quindi $\pi=\left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2$ ossia $\int_{-\infty}^{+\infty} e^{-x^2}dx=\sqrt{\pi}$ che NON si può ottenere integrando direttamente perchè le primitive di e^{-x^2} non sono esprimibili mediante funzioni elementari.