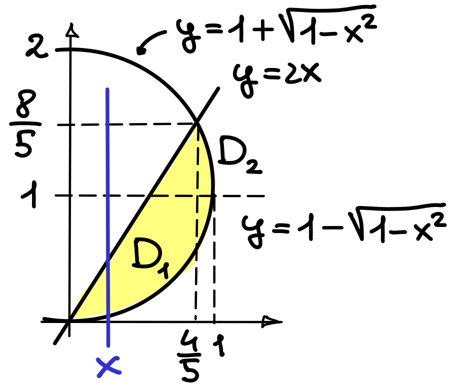
ANALISI MATEMATICA 2 - LEZIONE 18

ESEMPI

- $\iint_D x\,dx\,dy \qquad D=\{(x,y): 0\le y\le 2x,\ x^2+y^2\le 2y\}$

$x^2+(y-1)^2\le 1$

1) $D=D_1\cup D_2$ con D_1 e D_2 y-semplici



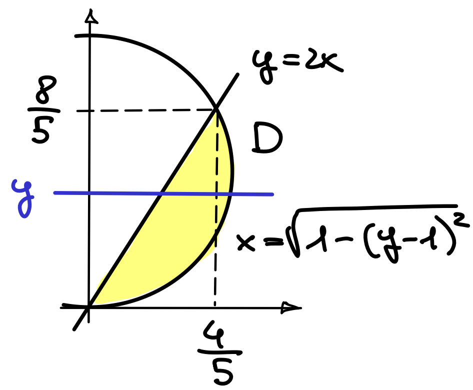
$$\iint_D x\,dx\,dy=\iint_{D_1} x\,dx\,dy+\iint_{D_2} x\,dx\,dy$$

$$=\int_{x=0}^{4/5}\left(\int_{y=1-\sqrt{1-x^2}}^{2x} x\,dy\right)dx+\int_{x=\frac{4}{5}}^{1}\left(\int_{y=1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} x\,dy\right)dx$$

$$=\int_{x=0}^{4/5} x\left(2x-1+\sqrt{1-x^2}\right)dx+\int_{x=4/5}^{1} 2x\sqrt{1-x^2}\,dx$$

$$=\left[\frac{2x^3}{3}-\frac{x^2}{2}-\frac{1}{3}(1-x^2)^{3/2}\right]_0^{4/5}+\left[-\frac{2}{3}(1-x^2)^{3/2}\right]_{4/5}^{1}=\dots=\frac{32}{75}.$$

2) D è x-semplice



$$\iint_D x\,dx\,dy=\int_{y=0}^{8/5}\left(\int_{x=\frac{y}{2}}^{\sqrt{1-(y-1)^2}} x\,dx\right)dy$$

$$=\int_{y=0}^{8/5}\left[\frac{x^2}{2}\right]_{\frac{y}{2}}^{\sqrt{1-(y-1)^2}}dy$$

$$=\frac{1}{2}\int_0^{8/5}\left(1-(y-1)^2-\frac{y^2}{4}\right)dy=\frac{1}{2}\int_0^{8/5}\left(2y-\frac{5}{4}y^2\right)dy$$

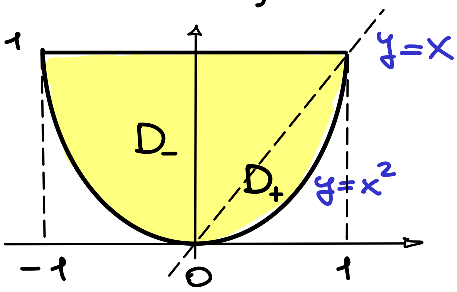
$$=\frac{1}{2}\left[y^2-\frac{5}{12}y^3\right]_0^{8/5}=\frac{1}{\cancel{2}}\cdot\frac{\overset{32}{\cancel{64}}}{25}\left(1-\frac{\cancel{5}}{\underset{3}{\cancel{12}}}\cdot\frac{\overset{2}{\cancel{8}}}{\cancel{5}}\right)=\frac{32}{75}.$$

- $\iint_D |x-y|\,dx\,dy \qquad D=\{(x,y): x^2\le y\le 1\}$

$D = D_+ \cup D_-$ (D_+: $x-y\ge 0$, D_-: $x-y\le 0$)

D_+ è y-semplice

D_- è x-semplice



$$\iint_D |x-y|\,dx\,dy = \iint_{D_+}(x-y)\,dx\,dy + \iint_{D_-}(y-x)\,dx\,dy$$

$$= \int_{x=0}^{1}\left(\int_{y=x^2}^{x}(x-y)\,dy\right)dx + \int_{y=0}^{1}\left(\int_{x=-\sqrt{y}}^{y}(y-x)\,dx\right)dy$$

$$= \int_0^1 \left[xy-\frac{y^2}{2}\right]_{x^2}^{x}dx + \int_0^1\left[yx-\frac{x^2}{2}\right]_{-\sqrt{y}}^{y}dy$$

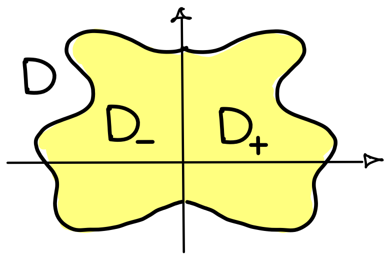
$$= \int_0^1\left(x^2-\frac{x^2}{2}-x^3+\frac{x^4}{2}\right)dx + \int_0^1\left(y^2-\frac{y^2}{2}+y^{3/2}+\frac{y}{2}\right)dy$$

$$= \left[\frac{x^3}{6}-\frac{x^4}{4}+\frac{x^5}{10}\right]_0^1 + \left[\frac{y^3}{6}+\frac{2}{5}y^{5/2}+\frac{y^2}{4}\right]_0^1$$

$$= \frac{1}{6}-\cancel{\frac{1}{4}}+\frac{1}{10}+\frac{1}{6}+\frac{2}{5}+\cancel{\frac{1}{4}} = \frac{10+6+10+24}{60} = \frac{5}{6}.$$

OSSERVAZIONE

Sia D un insieme simmetrico rispetto alla retta $x=0$:



dove $D_+=\{(x,y)\in D: x\ge 0\}$ e $D_-=\{(x,y)\in D: x\le 0\}$.

1) Se $f(x,y)=f(-x,y)$ ossia f è x-PARI allora

$$\iint_D f(x,y)\,dxdy = \iint_{D_+} f(x,y)\,dxdy + \iint_{D_-} f(x,y)\,dxdy$$

$$= \iint_{D_+} f(x,y)\,dxdy + \iint_{D_+} \underbrace{f(-x,y)}_{= f(x,y)}\,dxdy = 2\iint_{D_+} f(x,y)\,dxdy$$

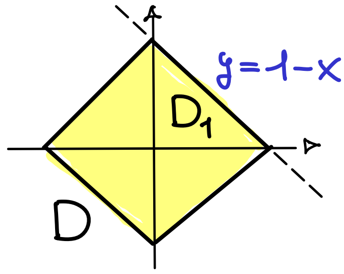
2) Se $f(x,y)=-f(-x,y)$ ossia f è x-DISPARI allora

$$\iint_D f(x,y)\,dxdy = \iint_{D_+} f(x,y)\,dxdy + \iint_{D_-} f(x,y)\,dxdy$$

$$= \iint_{D_+} f(x,y)\,dxdy + \iint_{D_+} \underbrace{f(-x,y)}_{= -f(x,y)}\,dxdy = 0$$

Analoghe relazioni valgono per domini simmetrici rispetto a $y=0$ e funzioni y-pari o y-dispari.

- $\displaystyle\iint_D x(x+e^{x^2})|y|\,dxdy$ $\qquad$ simmetrico rispetto a $x=0$ e $y=0$ $\quad D=\{(x,y):\ |x|+|y|\leq 1\}$



$$= \iint_D \underbrace{x^2|y|}_{\substack{\text{x-pari}\\ \text{y-pari}}}\,dxdy + \iint_D \underbrace{x e^{x^2}|y|}_{\substack{\text{x-dispari}\\ \text{(y-pari)}}}\,dxdy$$

$$= 4\iint_{D_1} x^2|y|\,dxdy + 0 \quad \text{dove } D_1=\{(x,y): x\geq 0, y\geq 0, x+y\leq 1\}$$

$$= 4\int_{x=0}^{1} x^2\left(\int_{y=0}^{1-x} y\,dy\right)dx = \cancel{4}^{\,2}\int_0^1 x^2\left[\frac{y^2}{\cancel{2}}\right]_0^{1-x} dx$$

$$= 2\int_0^1 x^2(1-x^2)\,dx = 2\left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{6-15+10}{15} = \frac{1}{15}.$$

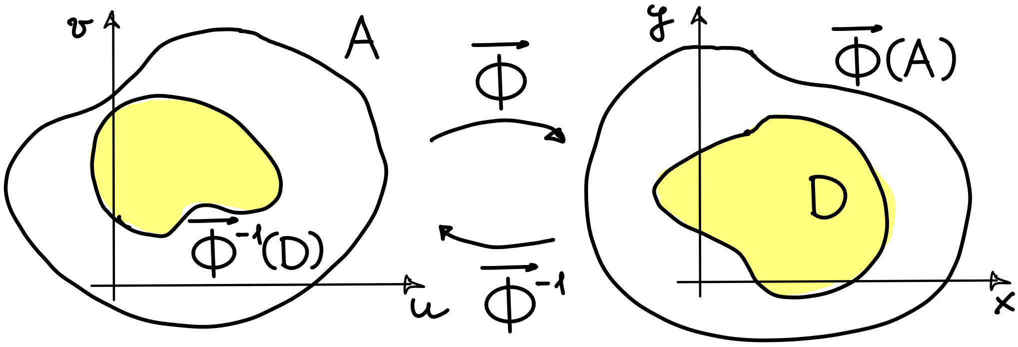
CAMBIAMENTO DI VARIABILI PER INTEGRALI DOPPI

Alle volte in un integrale doppio le variabili "originali" (x,y) possono non agevolare il calcolo. Può essere allora utile effettuare un CAMBIAMENTO DI VARIABILI passando ad un nuovo sistema di coordinate (u,v).

Indichiamo con $\vec{\Phi}: A \to \vec{\Phi}(A)$

$$\vec{\Phi}(u,v) = (\overbrace{\varphi_1(u,v)}^{x}, \overbrace{\varphi_2(u,v)}^{y})$$

la funzione biunivoca che realizza il cambio dove A è un insieme aperto di $\mathbb{R}^2$.



con $x = \varphi_1(u,v)$ e $y = \varphi_2(u,v)$.

Supponiamo che $\varphi_1, \varphi_2 \in C^1(A)$ e consideriamo la MATRICE JACOBIANA di $\vec{\Phi}$.

$$J_{\vec{\Phi}}(u,v) = \begin{bmatrix} \frac{\partial \varphi_1}{\partial u}(u,v) & \frac{\partial \varphi_1}{\partial v}(u,v) \\ \frac{\partial \varphi_2}{\partial u}(u,v) & \frac{\partial \varphi_2}{\partial v}(u,v) \end{bmatrix}$$

$\boxed{\text{TEOREMA}}$ (CAMBIO DI VARIABILI PER INTEGRALI DOPPI)

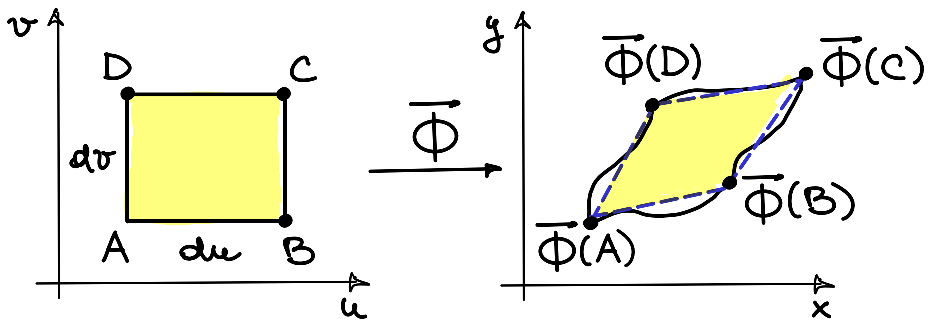
Sia $\vec{\Phi}: A \to \vec{\Phi}(A)$ un cambio di variabili nelle ipotesi descritte sopra. Sia D un insieme limitato e misurabile contenuto in $\vec{\Phi}(A)$ e sia f una funzione continua e limitata in D.

Se $\det(J_{\vec{\Phi}}(u,v)) \neq 0$ in $\vec{\Phi}^{-1}(D) \setminus E$ con $|E|=0$ allora

$$\iint_D f(x,y)\,dx\,dy = \iint_{\vec{\Phi}^{-1}(D)} f(\vec{\Phi}(u,v)) \left|\det(J_{\vec{\Phi}}(u,v))\right| du\,dv.$$

OSSERVAZIONE

Il termine $|\det(J_{\vec{\Phi}}(u,v))|$ rappresenta il fattore di trasformazione dell'elemento infinitesimo d'area $du\,dv$.



Aree: $du\,dv \xrightarrow{\vec{\Phi}} |\det(J_{\vec{\Phi}}(u,v))|\,du\,dv$

approssimazione lineare

Infatti dato che

$$\vec{\Phi}(B) - \vec{\Phi}(A) = \vec{\Phi}(u+du,v) - \vec{\Phi}(u,v)$$
$$\sim \left(\frac{\partial \varphi_1}{\partial u}(u,v), \frac{\partial \varphi_2}{\partial u}(u,v)\right) du$$

$$\vec{\Phi}(D) - \vec{\Phi}(A) = \vec{\Phi}(u,v+dv) - \vec{\Phi}(u,v)$$
$$\sim \left(\frac{\partial \varphi_1}{\partial v}(u,v), \frac{\partial \varphi_2}{\partial v}(u,v)\right) dv$$

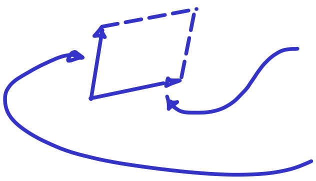
$$\vec{\Phi}(C)-\vec{\Phi}(D)=\vec{\Phi}(u+du,v+dv)-\vec{\Phi}(u,v+dv)$$
$$\sim \vec{\Phi}(B)-\vec{\Phi}(A)$$

$$\vec{\Phi}(C)-\vec{\Phi}(B)=\vec{\Phi}(u+du,v+dv)-\vec{\Phi}(u+du,v)$$
$$\sim \vec{\Phi}(D)-\vec{\Phi}(A)$$

Quindi

$$\text{Area}(\vec{\Phi}(ABCD))$$

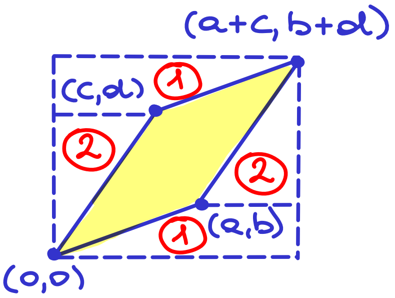
$\sim$ Area(parallelogramma con lati



$$\left(\frac{\partial\varphi_1}{\partial u}(u,v),\frac{\partial\varphi_2}{\partial u}(u,v)\right)du \quad \text{e}$$
$$\left(\frac{\partial\varphi_1}{\partial v}(u,v),\frac{\partial\varphi_2}{\partial v}(u,v)\right)dv \quad)$$

$$=\left|\det\begin{bmatrix}\frac{\partial\varphi_1}{\partial u}(u,v) & \frac{\partial\varphi_1}{\partial v}(u,v)\\ \frac{\partial\varphi_2}{\partial u}(u,v) & \frac{\partial\varphi_2}{\partial v}(u,v)\end{bmatrix}\right|du\,dv$$

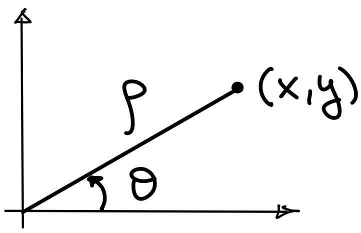
$$=\left|\det(J_{\vec{\Phi}}(u,v))\right|du\,dv.$$



Area(parallelogramma)

$$=(a+c)\cdot(b+d)-(2\,①+2\,②)$$
$$=(ab+cb+ad+cd)-(b(a+2c)+cd)$$
$$=ad-cb=\det\left(\begin{bmatrix}a & c\\ b & d\end{bmatrix}\right)$$

Un esempio importante di cambio di variabili in dimensione due è il passaggio dalle coordinate cartesiane alle COORDINATE POLARI (ρ,θ):



$$\begin{cases} x=\rho\cos\theta \\ y=\rho\,\text{sen}\,\theta.\end{cases}$$

$\rho=\sqrt{x^2+y^2}$

$\theta\in[0,2\pi)$

angolo principale

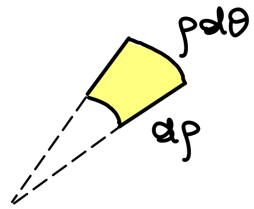
Poniamo $x=\varphi_1(\rho,\theta)=\rho\cos\theta$, $y=\varphi_2(\rho,\theta)=\rho\,\mathrm{sen}\,\theta$.
Allora la matrice jacobiana è

$$J_{\vec{\Phi}}(\rho,\theta)=\begin{bmatrix}\frac{\partial\varphi_1}{\partial\rho} & \frac{\partial\varphi_1}{\partial\theta}\\ \frac{\partial\varphi_2}{\partial\rho} & \frac{\partial\varphi_2}{\partial\theta}\end{bmatrix}=\begin{bmatrix}\cos\theta & -\rho\,\mathrm{sen}\,\theta\\ \mathrm{sen}\,\theta & \rho\cos\theta\end{bmatrix}$$

e dunque $|\det(J_{\vec{\Phi}}(\rho,\theta))|=\rho\cos^2\theta+\rho\,\mathrm{sen}^2\theta=\rho$. $=0$ solo nell'origine

L'elemento infinitesimo d'area è

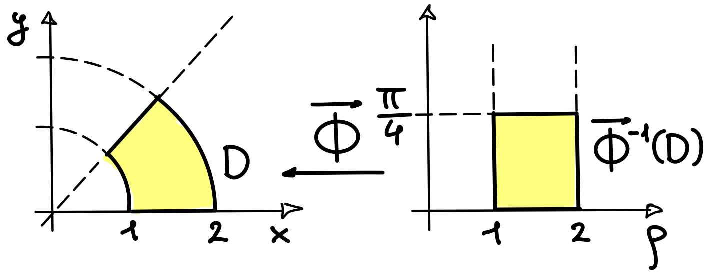
$$|\det(J_{\vec{\Phi}}(\rho,\theta))|\,d\rho\, d\theta=\rho\, d\rho\, d\theta.$$



ESEMPIO

- $\displaystyle\iint_D \frac{x}{(x^2+y^2)^2}\,dx\,dy \qquad D_1=\{(x,y): 0\le y\le x,\ 1\le x^2+y^2\le 4\}.$

$$\overset{CP}{=}\iint_{\vec{\Phi}^{-1}(D)} \frac{\cancel{\rho}\cos\theta}{\rho^{\cancel{4}\,2}}\cancel{\rho}\,d\rho\, d\theta$$



$$=\int_{\rho=1}^{2}\left(\frac{1}{\rho^2}\int_{\theta=0}^{\pi/4}\cos\theta\, d\theta\right)d\rho$$

$$\vec{\Phi}^{-1}(D)=\underbrace{[1,2]}_{\rho}\times\underbrace{\left[0,\frac{\pi}{4}\right]}_{\theta}$$

$$=\int_1^2\frac{1}{\rho^2}\Big[\mathrm{sen}\,\theta\Big]_0^{\pi/4}d\rho=\frac{1}{\sqrt2}\left[-\frac{1}{\rho}\right]_1^2=\frac{1}{\sqrt2}\left(-\frac12+1\right)=\frac{1}{2\sqrt2}.$$