ANALISI MATEMATICA 2 - LEZIONE 16

ALCUNI ESERCIZI DEL FOGLIO 4

1.a $2y - x^2 = 2 + \log(y)$ in $(0,1)$. T_2 di $y = \varphi(x)$ in $x = 0$?

Sia $f(x,y) = 2y - x^2 - 2 - \log(y)$.

$$f_y(0,1) = \left(2 - 2x - \frac{1}{y}\right)\Big|_{(0,1)} = 1 \neq 0 \overset{TFI}{\Longrightarrow} \exists\, y = \varphi(x).$$

$$\varphi(0) = 1,\ \varphi'(0) = -\frac{f_x(0,1)}{f_y(0,1)} = -\frac{(-2x)}{1}\Big|_{(0,1)} = 0.$$

Inoltre

$$2\varphi(x) - x^2 = 2 + \log(\varphi(x)) \overset{D}{\Longrightarrow} 2\varphi'(x) - 2x = \frac{\varphi'(x)}{\varphi(x)}$$

$$\overset{D}{\Longrightarrow} 2\varphi''(x) - 2 = \frac{\varphi''(x)\varphi(x) - (\varphi'(x))^2}{(\varphi(x))}$$

Per $x = 0$ otteniamo

$$2\varphi''(0) - 2 = \frac{\varphi''(0)\cdot 1 - 0^2}{1^2} \Longrightarrow \varphi''(0) = 2.$$

Quindi

$$T_2(x) = 1 + 0\cdot x + \frac{2}{2}x^2 = 1 + x^2.$$

2.a $\begin{cases} xyz = -2 \\ x^3 + y^3 + z^3 = 8 \end{cases}$ in $(-1,2,1)$. T_1 di $y = \varphi(x)$ e $z = \psi(x)$ in $x = -1$?

Siano $f(x,y,z) = xyz + 2 = 0$, $g(x,y,z) = x^3 + y^3 + z^3 - 8 = 0$.

$$J = \begin{bmatrix} f_y & f_z \\ g_y & g_z \end{bmatrix} = \begin{bmatrix} xz & xy \\ 3y^2 & 3z^2 \end{bmatrix} \overset{(-1,2,1)}{\Longrightarrow} \begin{bmatrix} -1 & -2 \\ 12 & 3 \end{bmatrix} \Longrightarrow \det(J) = 21 \neq 0$$

Quindi per TFI $\exists\, y = \varphi(x)$ e $\exists\, z = \psi(x)$: $\varphi(-1) = 2$, $\psi(-1) = 1$.

Inoltre derivando

$$\begin{cases} x\varphi(x)\psi(x) = -2 \\ x^3 + (\varphi(x))^3 + (\psi(x))^3 = 8 \end{cases}$$

si ottiene

$$\begin{cases} \varphi(x)\psi(x) + x\varphi'(x)\psi(x) + x\varphi(x)\psi'(x) = 0 \\ 3x^2 + 3(\varphi(x))^2\cdot\varphi'(x) + 3(\psi(x))^2\cdot\psi'(x) = 0 \end{cases}$$

Per $x=-1$ si ha

$$\begin{cases} 2\cdot 1 - \varphi'(-1)\cdot 1 - 2\psi'(-1) = 0 \\ 3 + 12\varphi'(-1) + 3\psi'(-1) = 0 \end{cases} \qquad \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \varphi'(-1) \\ \psi'(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

da cui

$$\varphi'(-1) = \frac{\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}} = \frac{4}{-7} \quad e \quad \psi'(-1) = \frac{\begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}} = \frac{-9}{-7} = \frac{9}{7}.$$

REGOLA DI CRAMER

Infine

$$\varphi)\ T_1(x) = 2 - \frac{4}{7}(x+1) \qquad \psi)\ T_1(x) = 1 + \frac{9}{7}(x+1).$$

3.a $\{(x,y,z) : xyz = -8\}$ Piano tangente in $(-2,1,4)$?

Sia $f(x,y,z) = xyz + 8$. Allora

$$\nabla f(x,y,z) = (yz, xz, xy)$$

e il piano tangente in $(-2,1,4)$ è

$$f_x(-2,1,4)(x+2) + f_y(-2,1,4)(y-1) + f_z(-2,1,4)(z-4) = 0$$

ossia $4(x+2) - 8(y-1) - 2(z-4) = 0$

$$2x - 4y - z + 12 = 0.$$

4.a $f(x,y)=xy+x^2$ Punti stazionari vincolati in $\Gamma=\{(x,y): x^2+y^2+xy=4\}$?

Sia $g(x,y)=x^2+y^2+xy-4=0$ allora

$\nabla g(x,y)=(2x+y,2y+x)=(0,0) \iff (x,y)=(0,0)\notin\Gamma$

Tutti i punti di Γ sono regolari

Punti stazionari vincolati

$$\begin{cases}\nabla f(x,y)=\lambda\nabla g(x,y)\\ g(x,y)=0\end{cases}\qquad \begin{cases} y+2x=\lambda(2x+y) \to (\lambda-1)(2x+y)=0 \\ x=\lambda(2y+x) \\ x^2+y^2+xy=4\end{cases}$$

$\lambda=1$ $\quad$ $y=-2x$

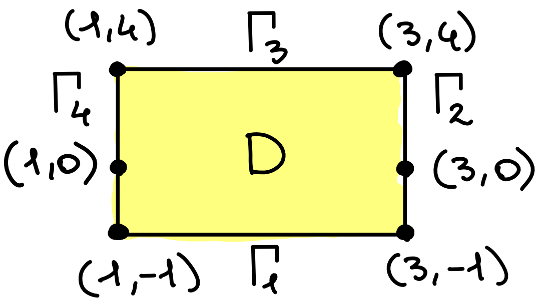
$$\begin{cases}\lambda=1\\ \cancel{x}=2y+\cancel{x} \to y=0\\ x^2+y^2+xy=4 \to x^2=4\to x=\pm 2\end{cases} \Rightarrow \boxed{\begin{matrix}(2,0)\\(-2,0)\end{matrix}} \quad \lambda=1$$

$$\begin{cases} y=-2x & \to\ y=\mp\frac{4}{\sqrt3}\\ x=\lambda(-4x+x) & \to\ \lambda=-\frac{1}{3}\\ x^2+4x^2-2x^2=4 & \to\ x=\pm\frac{2}{\sqrt3}\end{cases} \Rightarrow \boxed{\begin{matrix}\left(\frac{2}{\sqrt3},-\frac{4}{\sqrt3}\right)\\ \left(-\frac{2}{\sqrt3},\frac{4}{\sqrt3}\right)\end{matrix}} \quad \lambda=-\frac{1}{3}$$

$3x^2=4$

5.a $f(x,y)=\log((x+1)^2+y^2)$ Max/min in $D=[1,3]\times[-1,4]$?

$f\in C^2(\mathbb{R}^2\setminus\{(-1,0)\})$, D è compatto e $(-1,0)\notin D$.



$\partial D=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$

Punti stazionari interni a D:

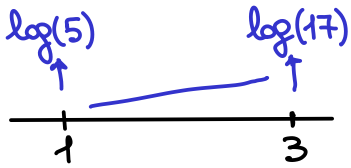
$$\nabla f(x,y)=\left(\frac{2(x+1)}{(x+1)^2+y^2},\frac{2y}{(x+1)^2+y^2}\right)=(0,0)$$

Senza soluzioni in $\mathbb{R}^2\setminus\{(-1,0)\}$

Bordo di D: $\partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$

1) $\Gamma_1 = \{(x,-1) : x \in [1,3]\}$,

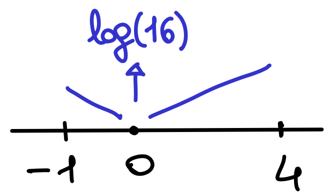
$h(x) = f(x,-1) = \log((x+1)^2+1)$



$$\boxed{f(1,-1) = \log(5)\ ,\ f(3,-1) = \log(17)}$$

2) $\Gamma_2 = \{(3,y) : y \in (-1,4)\}$,

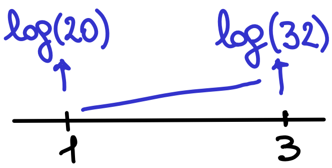
$h(y) = f(3,y) = \log(16+y^2)$



$$\boxed{f(3,0) = \log(16)}$$

3) $\Gamma_3 = \{(x,4) : x \in [1,3]\}$,

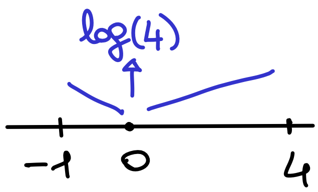
$h(x) = f(x,4) = \log((x+1)^2+16)$



$$\boxed{f(1,4) = \log(20)\ ,\ f(3,4) = \log(32)}$$

4) $\Gamma_4 = \{(1,y) : y \in (-1,4)\}$,

$h(y) = f(1,y) = \log(4+y^2)$



$$\boxed{f(1,0) = \log(4)}$$

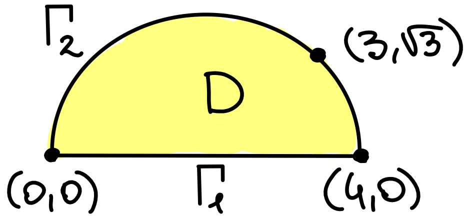
Confrontando i valori trovati si ha che $(1,0)$ è un punto di minimo assoluto con valore $\log(4)$ e $(3,4)$ è un punto di massimo assoluto con valore $\log(32)$.

$\boxed{5.d}$ $f(x,y) = x + \sqrt{3}y$ Max/min in

$D = \{(x,y) : x^2+y^2 \leq 4x,\ y \geq 0\}$?

$(x-2)^2 + y^2 \leq 4$

$f \in C^2(\mathbb{R}^2)$ e D è compatto



$\partial D = \Gamma_1 \cup \Gamma_2$

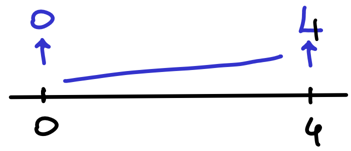
Non ci sono punti stazionari interni perché

$$\nabla f(x,y) = (1,\sqrt{3}) \neq (0,0)$$

Bordo di D: $\partial D = \Gamma_1 \cup \Gamma_2$ $\boxed{f(0,0)=0,\ f(4,0)=4}$

1) $\Gamma_1 = \{(x,0) : x \in [0,4]\}$

$h(x) = f(x,0) = x$



2) $\Gamma_2 = \{(x,y) : \underbrace{x^2+y^2-4x}_{= g(x,y)} = 0,\ y>0\}$

$\nabla g(x,y) = (2x-4, 2y) = (0,0) \Rightarrow (x,y) = (2,0) \notin \Gamma_2$ tutti regolari

Punti stazionari vincolati: $\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$

$$\begin{cases} 1 = \lambda(2x-4) \\ \sqrt{3} = \lambda(2y) \\ x^2+y^2 = 4x \end{cases} \begin{array}{l} \rightarrow y = \sqrt{3}(x-2) \\ \rightarrow \lambda = \frac{\sqrt{3}}{2y} \\ \rightarrow x^2 + 3(x-2)^2 = 4x,\ x^2+3x^2-12x+12 = 4x \end{array}$$

$$x^2 - 4x + 3 = 0 \rightarrow x = 2 \pm 1 = \begin{cases} 1 \\ 3 \end{cases}$$

$$\begin{cases} x = 1 \\ y = -\sqrt{3} \\ \lambda = -\frac{1}{2} \end{cases} \cup \begin{cases} x = 3 \\ y = \sqrt{3} \\ \lambda = \frac{1}{2} \end{cases}$$

$(1,-\sqrt{3}) \notin \Gamma_2$

$(3,\sqrt{3}) \in \Gamma_2$ $\boxed{f(3,\sqrt{3}) = 6}$

Confrontando i valori trovati si ha che $(0,0)$ è un punto di minimo assoluto con valore 0 e $(3,\sqrt{3})$ è un punto di massimo assoluto con valore 6.