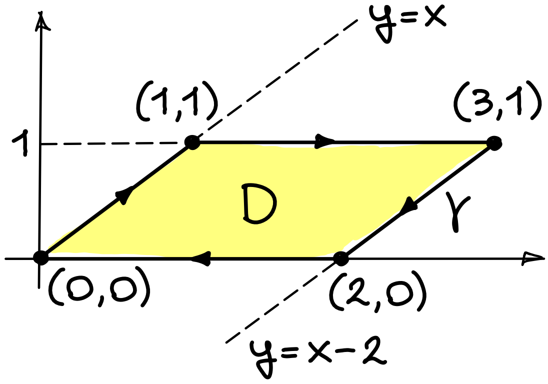
# ANALISI MATEMATICA 2 - FOGLIO 7

**1.a** Calcolare $\int_\gamma \langle \vec{F}, d\vec{s} \rangle$ dove $\vec{F} = ((x^2+y^2)e^x, ye^x)$

e $\gamma$ è il quadrilatero di vertici $(0,0), (2,0), (3,1), (1,1)$ percorso in senso orario.




Il percorso $\gamma$ è il bordo di

$$D = \{(x,y) : y \le x \le y+2, y \in [0,1]\}$$

e dunque per la formula di Gauss-Green

$$\int_\gamma \langle \vec{F}, d\vec{s} \rangle \underset{\text{orario}}{=} -\int_{\partial^+ D} \langle \vec{F}, d\vec{s} \rangle \overset{GG}{=} -\iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

$$= -\iint_D (ye^x - 2ye^x)\,dx\,dy = \int_{y=0}^{1} y \left( \int_{x=y}^{y+2} e^x\,dx \right) dy$$

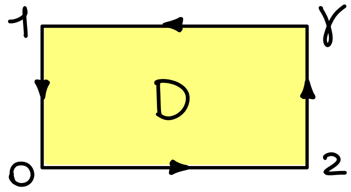
$$= \int_0^1 y(e^{y+2} - e^y)\,dy = (e^2-1)\int_0^1 ye^y\,dy$$

$$= (e^2-1)\left[e^y(y-1)\right]_0^1 = e^2 - 1.$$

**1.b** Calcolare $\int_\gamma \langle \vec{F}, d\vec{s}\rangle$ dove

$(1+x)(1+y)$

$$\vec{F} = \left(\frac{x+3y}{(1+x)^2},\ 3y^2\log(1+x+y+xy)\right)$$




e $\gamma$ è il bordo di $D=[0,2]\times[0,1]$ percorso in senso antiorario.

Notiamo che il dominio di $\vec{F}$ contiene $D$ e possiamo applicare la formula di Gauss-Green

$$\int_\gamma \langle \vec{F}, d\vec{s}\rangle = \int_{\partial^+ D} \langle \vec{F}, d\vec{s}\rangle \overset{GG}{=} \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

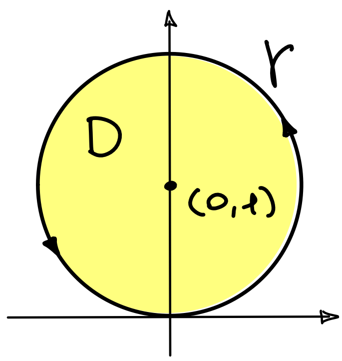
$$= \int_{x=0}^{2}\int_{y=0}^{1} \left(\frac{3y^2}{1+x} - \frac{3}{(1+x)^2}\right) dx\,dy$$

$$= \int_{x=0}^{2} \left(\frac{\cancel{3}}{1+x}\left[\frac{y^3}{\cancel{3}}\right]_0^1 - \frac{3}{(1+x)^2}\left[y\right]_0^1\right) dx$$

$$= \left[\log(1+x) + \frac{3}{1+x}\right]_0^2 = \log 3 + 1 - 3 = \log 3 - 2.$$

**1.c** Calcolare $\int_\gamma \langle \vec{F}, d\vec{s}\rangle$ dove

$$\vec{F} = \left(-7yx^2,\ (\sqrt{y}+x)y^2\right)$$




e $\gamma$ è il bordo di

$x^2+(y-1)^2 \le 1$

$$D = \left\{(x,y) : x^2+y^2 \le 2y\right\}$$

percorso in senso antiorario.

$$\int_\gamma \langle \vec{F}, d\vec{s}\rangle = \int_{\partial^+ D} \langle \vec{F}, d\vec{s}\rangle \overset{GG}{=} \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$x=\rho\cos\theta \qquad y=1+\rho\,\mathrm{sen}\,\theta$$

$$=\iint_D (y^2+7x^2)\,dx\,dy \overset{CP}{=} \int_{\theta=0}^{2\pi}\int_{\rho=0}^{1}\left((1+\rho\,\mathrm{sen}\,\theta)^2+7(\rho\cos\theta)^2\right)\rho\,d\rho\,d\theta$$

$$=\int_0^{2\pi}\left(\int_0^1\Big(\rho+\underbrace{2\rho^2\mathrm{sen}\,\theta}_{\theta\text{-dispari}}+\overbrace{\rho^3\mathrm{sen}^2\theta+7\rho^3\cos^2\theta}^{\rho^3(1+6\cos^2\theta)}\Big)d\rho\right)d\theta$$

$$=\int_0^{2\pi}\left[\frac{\rho^2}{2}+\frac{\rho^4}{4}(1+6\cos^2\theta)\right]_0^1 d\theta=\int_0^{2\pi}\left(\frac{3}{4}+\frac{3}{2}\cos^2\theta\right)d\theta$$

$$=\frac{3}{4}\cdot 2\pi+\frac{3}{2}\cdot\pi=3\pi.$$

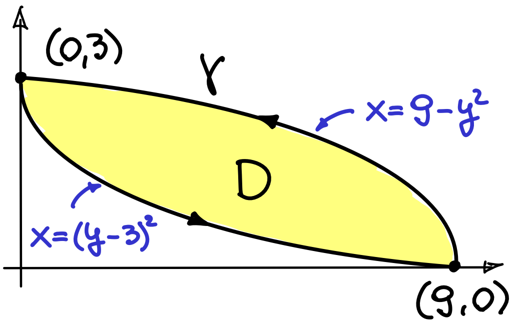
$\boxed{1.d}$ Calcolare $\int_\gamma \langle \vec{F}, d\vec{s}\rangle$ dove

$$\vec{F}=(\mathrm{sen}\,x+2y,\ (x+1)^2+\cos y)$$

dove $\gamma$ è il bordo di

$$D=\{(x,y):(y-3)^2\le x\le 9-y^2\}$$

percorso in senso antiorario.




$$\int_\gamma\langle\vec{F},d\vec{s}\rangle=\int_{\partial^+D}\langle\vec{F},d\vec{s}\rangle\overset{GG}{=}\iint_D\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dx\,dy$$

$$=\iint_D\big(2(x+1)-2\big)\,dx\,dy=\int_{y=0}^{3}\left(\int_{(y-3)^2}^{9-y^2}2x\,dx\right)dy$$

$$=\int_0^3\left((9-y^2)^2-(y-3)^4\right)dy$$

$$\overset{t=3-y}{=}\int_0^3\Big(\underbrace{(9-(3-t)^2)^2}_{(\cancel{9}-\cancel{9}+6t-t^2)^2=36t^2-12t^3+\cancel{t^4}}-\cancel{t^4}\Big)dt=\left[12t^3-3t^4\right]_0^3=81(4-3)=81.$$

**1.e** Calcolare $\int_\gamma \langle \vec{F}, d\vec{s} \rangle$ dove

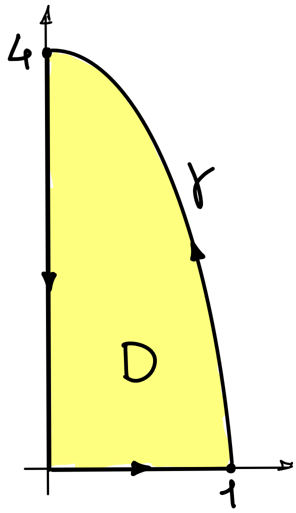
$$\vec{F} = \left(e^{-x^2} - y,\ 4x^3 + \log(1+y)\right)$$

$\log(1+y)$: $1+y > 0$ in D

dove $\gamma$ è il bordo di

$$D = \{(x,y) : 16x^2 + y^2 \leq 16,\ x \geq 0, y \geq 0\}$$

percorso in senso antiorario.




$$\int_\gamma \langle \vec{F}, d\vec{s} \rangle = \int_{\partial^+ D} \langle \vec{F}, d\vec{s} \rangle \overset{GG}{=} \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \iint_D (12x^2 + 1)\, dx\,dy \overset{\begin{cases} x = u \\ y = 4v \end{cases}}{=} \iint_{\left\{\substack{u^2+v^2 \leq 1 \\ u,v \geq 0}\right\}} (12u^2+1) \cdot 4\, du\,dv$$

$$\overset{CP}{=} 4 \cdot 12 \int_{\rho=0}^{1} \int_{\theta=0}^{\pi/2} \rho^2 \cos^2\theta\, \rho\, d\rho\, d\theta + 4 \cdot \frac{\pi}{4}$$

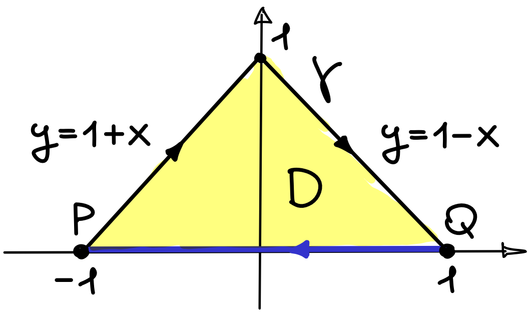
$$= \cancel{4} \cdot \overset{3}{\cancel{12}} \cdot \frac{\pi}{\cancel{4}} \cdot \left[\frac{\rho^4}{\cancel{4}}\right]_0^1 + \pi = 3\pi + \pi = 4\pi.$$

**1.f** Calcolare $\int_\gamma \langle \vec{F}, d\vec{s} \rangle$ dove

$$\vec{F} = \left(\frac{x + y^2}{1 + x^2},\ y^2 + y \arctg(x)\right)$$

e $\gamma$ è l'unione dei segmenti da $(-1,0)$ a $(0,1)$ e da $(0,1)$ a $(1,0)$.

In alternativa al calcolo diretto possiamo usare la formula di GG chiudendo $\gamma$ con il segmento da Q a P.



$$\int_\gamma \langle \vec{F}, d\vec{s}\rangle \overset{GG}{=} \underset{\text{orario}}{\uparrow -} \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy - \int_{[Q,P]} \langle \vec{F}, d\vec{s}\rangle$$

$$= -\iint_D \left(\frac{y}{1+x^2} - \frac{2y}{1+x^2}\right) dx\,dy + \int_{-1}^{1} F_1(t,0)\,dt$$

$$= \overset{\text{x-pari}}{\iint_D \frac{y}{1+x^2}\,dx\,dy} + \overset{\text{dispari}}{\int_{-1}^{1} \frac{t}{1+t^2}\,dt}$$

simmetrico per $x=0$ →

$$= 2\int_{x=0}^{1} \frac{1}{1+x^2}\left(\int_{y=0}^{1-x} y\,dy\right)dx + 0 = \cancel{2}\int_0^1 \frac{1}{1+x^2}\left[\frac{y^2}{\cancel{2}}\right]_0^{1-x} dx$$

$$= \int_0^1 \frac{\overbrace{1+x^2-2x}^{1-\frac{2x}{1+x^2}}}{1+x^2}\,dx = 1 - \left[\log(1+x^2)\right]_0^1 = 1 - \log 2.$$

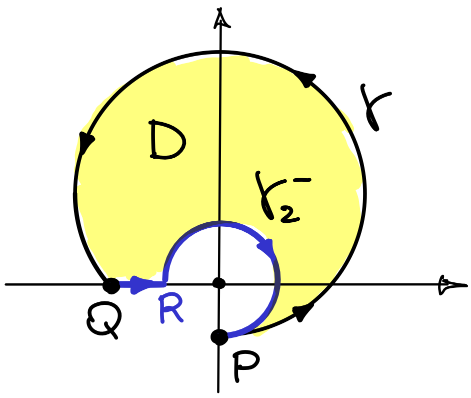
$\boxed{1.9}$ Calcolare $\int_\gamma \langle \vec{F}, d\vec{s}\rangle$ dove $\vec{F} = \left(\frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2}\right)$

e $\vec{\gamma}(t) = (\sqrt{2}\cos t, 1+\sqrt{2}\sin t)$, $t \in \left[-\frac{\pi}{2}, \frac{5\pi}{4}\right]$.

$\gamma$ è l'arco di una circonferenza centrata in $(0,1)$ e raggio $\sqrt{2}$.

$P = \vec{\gamma}(-\frac{\pi}{2}) = (0, 1-\sqrt{2})$

$Q = \vec{\gamma}(\frac{5\pi}{4}) = (-1, 0)$




Consideriamo $\vec{F} = \vec{F}_1 + \vec{F}_2$ con

$\vec{F}_1 = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ conservativo in $\mathbb{R}^2 \setminus \{(0,0)\}$

e con potenziale $U_1(x,y) = \log(\sqrt{x^2+y^2})$

$\vec{F}_2 = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ irrotazionale in $\mathbb{R}^2 \setminus \{(0,0)\}$.

Allora

$$\int_\gamma \langle \vec{F_1}, d\vec{s} \rangle = U_1(Q) - U_1(P) = \underbrace{\log(1)}_{=0} - \log(\sqrt{2}-1).$$

Per $\vec{F_2}$ chiudiamo $\gamma$ in modo che il percorso non si avvolga intorno a $(0,0)$ e che sia composto da segmenti radiali e archi di circonferenze centrate in $(0,0)$ (così il calcolo è più agevole).

Ad esempio consideriamo D tale che

$$\partial^+ D = \gamma \cup [Q,R] \cup \gamma_2^-$$

con $R = (1-\sqrt{2}, 0)$ e

$$\gamma_2(t) = (\sqrt{2}-1) \cdot (\cos t, \operatorname{sen} t) \quad \text{per } t \in \left[-\frac{\pi}{2}, \pi\right].$$

Così dato che $\vec{F_2}$ è irrotazionale in D per GG,

$$\int_\gamma \langle \vec{F_2}, d\vec{s} \rangle \overset{GG}{=} \iint_D 0\, dx dy - \int_{[Q,R] \cup \gamma_2^-} \langle \vec{F_2}, d\vec{s} \rangle$$

$$= -\underbrace{\int_{[Q,R]} \langle \vec{F_2}, d\vec{s} \rangle}_{=0} + \int_{\gamma_2} \langle \vec{F_2}, d\vec{s} \rangle$$

$$= \int_{-\frac{\pi}{2}}^{\pi} \frac{\cancel{(\sqrt{2}-1)^2}}{\cancel{(\sqrt{2}-1)^2}}\, dt = \pi - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{2}.$$

Si conclude che

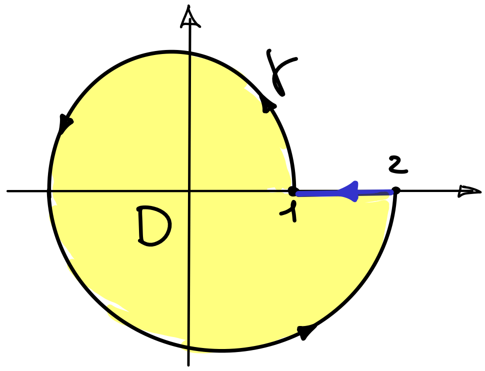
$$\int_\gamma \langle \vec{F}, d\vec{s} \rangle = \int_\gamma \langle \vec{F_2}, d\vec{s} \rangle + \int_\gamma \langle \vec{F_1}, d\vec{s} \rangle$$

$$= -\log(\sqrt{2}-1) + \frac{3\pi}{2}.$$

$\boxed{1.h}$ Calcolare $\int_\gamma \langle \vec{F}, d\vec{s} \rangle$ dove

$$\vec{F} = \left(3y + \frac{8x}{(x^2+y^2)^2}, \frac{8y}{(x^2+y^2)^2}\right)$$

e $\vec{\gamma}(t) = (1+t)\cdot(\cos(2\pi t), \mathrm{sen}(2\pi t))$, $t \in [0,1]$.




Si noti che $\|\vec{\gamma}(t)\| = 1 + t > 0$ per $t \in [0,1]$ e quindi la curva $\gamma$ non passa per $(0,0)$ dove $\vec{F}$ non è definito.

Poniamo $\vec{F} = \vec{F}_1 + \vec{F}_2$ dove $\vec{F}_1 = (3y, 0)$ e $\vec{F}_2 = \left(\frac{8x}{(x^2+y^2)^2}, \frac{8y}{(x^2+y^2)^2}\right)$. $\vec{F}_2$ è conservativo con potenziale $U_2(x,y) = \frac{-4}{x^2+y^2}$ per $(x,y) \neq (0,0)$.

Quindi

$$\int_\gamma \langle \vec{F}_2, d\vec{s} \rangle = U_2(\overset{(2,0)}{\overset{\parallel}{\vec{\gamma}(1)}}) - U_2(\overset{(1,0)}{\overset{\parallel}{\vec{\gamma}(0)}}) = \frac{-4}{2^2+0^2} + \frac{4}{1^2+0^2} = 3.$$

Per $\vec{F}_1$ in alternativa al calcolo diretto applichiamo GG chiudendo il percorso con il segmento da $(2,0)$ a $(0,1)$.

$$\int_\gamma \langle \vec{F}_1, d\vec{s} \rangle = \iint_D \left(\frac{\partial(0)}{\partial x} - \frac{\partial(3y)}{\partial y}\right) dx\,dy - \int_{[2,1]} \langle (3\overset{0}{\overset{\parallel}{y}}, 0), d\vec{s} \rangle$$

$$= -3|D| - 0 = -3 \cdot \frac{1}{2} \int_0^{2\pi} \rho^2(\theta)\, d\theta$$

$$\rho(\theta) = 1 + \frac{\theta}{2\pi} \quad = -\frac{\cancel{3}}{\cancel{2}} \left[\frac{1}{\cancel{3}}\left(1+\frac{\theta}{2\pi}\right)^3 \cdot \cancel{2}\pi\right]_0^{2\pi} = -\pi(8-1) = -7\pi.$$

Così

$$\int_\gamma \langle \vec{F}, d\vec{s} \rangle = \int_\gamma \langle \vec{F}_1, d\vec{s} \rangle + \int_\gamma \langle \vec{F}_2, d\vec{s} \rangle = -7\pi + 3.$$

$\boxed{1.i}$ Calcolare $\int_\gamma \langle \vec{F}, d\vec{s} \rangle$ dove

$$\vec{F} = (6y^2 + y^2 e^x - 4y, 2ye^x)$$

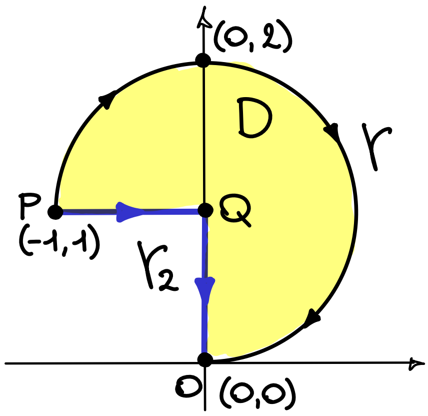
e $\gamma$ è l'arco di circonferenza da $(-1,1)$ a $(0,0)$ passante per $(0,2)$.

Intanto osserviamo che

$$\vec{F} = \nabla U + (6y^2 - 4y, 0) \quad \text{con } U(x,y) = y^2 e^x$$

Consideriamo anche la spezzata $\vec{\gamma}_2 = [P,Q] \cup [Q,O]$ con

$$[P,Q] : [-1,0] \ni t \to (t,1)$$
$$[Q,O] : [1,0] \ni t \to (0,t)$$




Applicando GG a D si ha

$$\int_\gamma \langle \vec{F}, d\vec{s} \rangle \overset{GG}{=} -\iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy + \int_{\gamma_2} \langle \vec{F}, d\vec{s} \rangle$$
$$= (6\pi + 4) + (2 - e^{-1}) = 6\pi + 6 - e^{-1}$$

perché

$$\iint_D \left( \frac{\partial F_2}{\partial x} - \underbrace{\frac{\partial F_1}{\partial y}}_{\frac{\partial}{\partial y}(6y^2-4y)} \right) dxdy = \iint_D (-12y + 4) dxdy$$

$$\begin{cases} x=\rho\cos\theta \\ y=1+\rho\,\mathrm{sen}\,\theta \end{cases} \quad = \int_{\theta=-\frac{\pi}{2}}^{\pi}\int_{\rho=0}^{1}(-12-12\rho\,\mathrm{sen}\,\theta+4)\,\rho\,d\rho\,d\theta$$

$$= -8\cdot\frac{3\pi}{2}\cdot\left[\frac{\rho^2}{2}\right]_0^1 + 12\left[\cos\theta\right]_{-\frac{\pi}{2}}^{\pi}\cdot\left[\frac{\rho^3}{3}\right]_0^1$$

$$= -6\pi + 12(-1)\cdot\frac{1}{3} = -6\pi - 4$$

e

$$\int_{\gamma_2}\langle\vec{F},d\vec{s}\rangle = U(0,0)-U(-1,1) + \int_{[P,Q]\cup[Q,O]}\langle(6y^2-4y,0),d\vec{s}\rangle$$

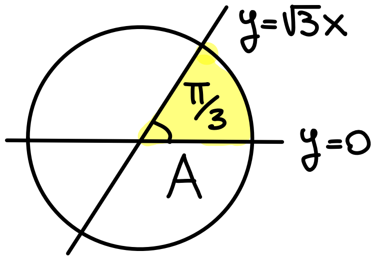
$$= 0-e^{-1}+\int_{-1}^{0}\langle(2,0),(1,0)\rangle\,dt + \int_{1}^{0}\overbrace{\langle(6t^2-4t,0),(0,1)\rangle}^{0}\,dt$$

$$= 2-e^{-1}.$$

**2.a** Calcolare $\displaystyle\iint_S z\,dS$ con

$$S=\{(x,y,z):\ z=xy,\ x^2+y^2\le 1,\ 0\le y\le \sqrt{3}x\}.$$

Parametrizzazione di $S$:




$$\vec{\sigma}(x,y)=(x,y,xy)$$

con $A=\{(x,y):x^2+y^2\le 1,\ 0\le y\le \sqrt{3}x\}$.

Quindi

$$\vec{\sigma}_x\times\vec{\sigma}_y=\left(-\frac{\partial(xy)}{\partial x},-\frac{\partial(xy)}{\partial y},1\right)=(-y,-x,1).$$

Allora

$$\iint_S z\,dS=\iint_A xy\,\|\vec{\sigma}_x\times\vec{\sigma}_y\|\,dx\,dy$$

$$=\iint_A xy\sqrt{1+x^2+y^2}\,dx\,dy$$

$$\overset{CP}{=}\int_{\theta=0}^{\pi/3}\int_{\rho=0}^{1}\rho^2\cos\theta\,\mathrm{sen}\,\theta\cdot\sqrt{1+\rho^2}\cdot\rho\,d\rho\,d\theta$$

$t=1+\rho^2$
$dt=2\rho\,d\rho$

$$=\left[\frac{\mathrm{sen}^2\theta}{2}\right]_0^{\pi/3}\int_0^1\rho^3\cdot\sqrt{1+\rho^2}\,d\rho\overset{\downarrow}{=}\frac{(\sqrt{3}/2)^2}{2}\int_1^2(t-1)\sqrt{t}\,\frac{dt}{2}$$

$$=\frac{3}{\cancel{16}_{\,8}}\left[\frac{\cancel{2}t^{5/2}}{5}-\frac{\cancel{2}t^{3/2}}{3}\right]_1^2=\frac{3}{8}\left(\sqrt{2}\left(\frac{4}{5}-\frac{2}{3}\right)-\left(\frac{1}{5}-\frac{1}{3}\right)\right)$$

$$=\frac{1}{40}\left(\sqrt{2}\cdot 2+2\right)=\frac{\sqrt{2}+1}{20}.$$

$\boxed{2.b}$ Calcolare $\displaystyle\iint_S \frac{x^2}{z^2}\,dS$ con

$$S=\{(x,y,z):(x^2+y^2)z^2=1,\ z\geq 0,\ 1\leq x^2+y^2\leq 3\}.$$

$S$ è il grafico di $z=f(x,y)=\dfrac{1}{\sqrt{x^2+y^2}}$ con $(x,y)$ in

$$A=\{(x,y):1\leq x^2+y^2\leq 3\}.$$

Parametrizzazione cartesiana di $S$:

$$\vec{\sigma}(x,y)=\left(x,y,\frac{1}{\sqrt{x^2+y^2}}\right) \text{ con } (x,y)\in A.$$

Quindi

$$\vec{\sigma}_x\times\vec{\sigma}_y=(-f_x,-f_y,1)=\left(\frac{x}{(x^2+y^2)^{3/2}},\frac{y}{(x^2+y^2)^{3/2}},1\right)$$

e

$$\|\vec{\sigma}_x\times\vec{\sigma}_y\|=\frac{\left(x^2+y^2+(x^2+y^2)^3\right)^{1/2}}{(x^2+y^2)^{3/2}}=\frac{\left(1+(x^2+y^2)^2\right)^{1/2}}{x^2+y^2}.$$

Infine

$$\iint_S\frac{x^2}{z^2}\,dS=\iint_A x^2\cancel{(x^2+y^2)}\cdot\frac{\left(1+(x^2+y^2)^2\right)^{1/2}}{\cancel{x^2+y^2}}\,dx\,dy$$

$$\overset{CP}{=}\int_{\theta=0}^{2\pi}\int_{\rho=1}^{\sqrt{3}}\rho^2\cos^2\theta\,(1+\rho^4)^{1/2}\rho\,d\rho\,d\theta=\pi\int_1^{\sqrt{3}}\rho^3(1+\rho^4)^{1/2}\,d\rho$$

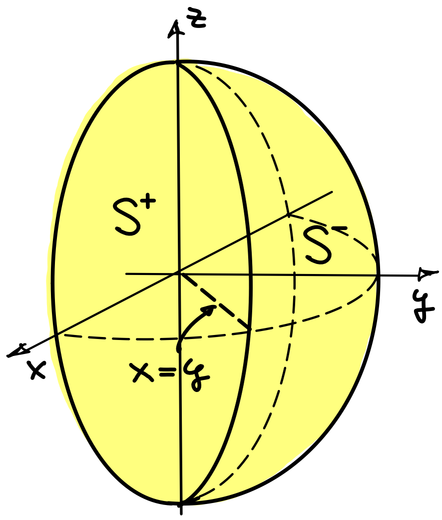
$t=1+\rho^4$
$dt=4\rho^3d\rho$

$$=\pi\int_2^{10}\sqrt{t}\,\frac{dt}{4}=\frac{\pi}{4}\left[\frac{2t^{3/2}}{3}\right]_2^{10}=\frac{\pi}{6}\left(10\sqrt{10}-2\sqrt{2}\right)$$

$$=\frac{\pi\sqrt{2}}{3}\left(5\sqrt{5}-1\right).$$

**2.C** Calcolare $\displaystyle\iint_S |x-y|\,dS$ con

$$S=\{(x,y,z): y\geq 0,\ x^2+y^2+z^2=4\}.$$




La semisfera $S$ è divisa in due parti dal piano $x=y$:

$$S^+ \text{ se } x\geq y \quad e \quad S^- \text{ se } x\leq y.$$

Con la parametrizzazione in coordinate sferiche

$$\vec{\sigma}(\theta,\varphi)=2(\cos\theta\,\mathrm{sen}\varphi, \mathrm{sen}\theta\,\mathrm{sen}\varphi, \cos\varphi)$$

si ha che $S^+=\vec{\sigma}(A^+)$ e $S^-=\vec{\sigma}(A^-)$ dove

$$A^+=\left[0,\tfrac{\pi}{4}\right]\times[0,\pi] \quad e \quad A^-=\left[\tfrac{\pi}{4},\pi\right]\times[0,\pi].$$

Dato che $\|\vec{\sigma}_\theta\times\vec{\sigma}_\varphi\|=4\,\mathrm{sen}\varphi$ si ottiene

$$\iint_S |x-y|\,dS=\iint_{S^+}(x-y)\,dS+\iint_{S^-}(y-x)\,dS$$

$$=\iint_{A^+}2(\cos\theta-\mathrm{sen}\theta)\,\mathrm{sen}\varphi\cdot 4\,\mathrm{sen}\varphi\,d\theta\,d\varphi$$

$$+\iint_{A^-}2(\mathrm{sen}\theta-\cos\theta)\,\mathrm{sen}\varphi\cdot 4\,\mathrm{sen}\varphi\,d\theta\,d\varphi$$

$$=8\int_0^\pi \mathrm{sen}^2\varphi\,d\varphi\cdot\left(\int_0^{\pi/4}(\cos\theta-\mathrm{sen}\theta)\,d\theta-\int_{\pi/4}^{\pi}(\cos\theta-\mathrm{sen}\theta)\,d\theta\right)$$

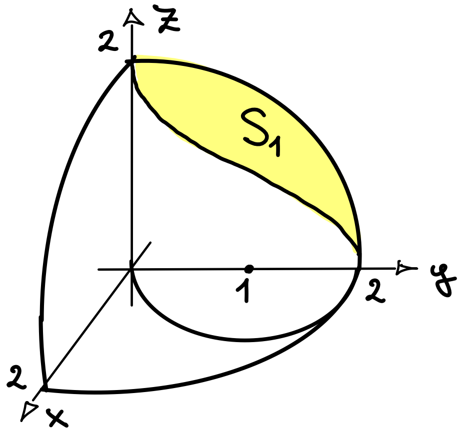
$$=8\cdot\frac{\pi}{2}\left(\Big[\mathrm{sen}\theta+\cos\theta\Big]_0^{\pi/4}-\Big[\mathrm{sen}\theta+\cos\theta\Big]_{\pi/4}^{\pi}\right)$$

$$=4\pi(\sqrt{2}-1+1+\sqrt{2})=8\sqrt{2}\,\pi.$$

**2.d** Calcolare l'area di $\quad x^2+(y-1)^2\leq 1$

$$S=\{(x,y,z): x^2+y^2+z^2=4,\ x^2+y^2\leq 2y\}.$$

$S$ è la parte della sfera $x^2+y^2+z^2=4$ contenuta nel cilindro $x^2+(y-1)^2\leq 1$.




Dato che $S$ è simmetrica rispetto ai piani $x=0$ e $z=0$ basta calcolare 4 volte l'area di $S_1$ ossia la parte di $S$ contenuta nel primo ottante.

Svolgiamo il calcolo in due modi.

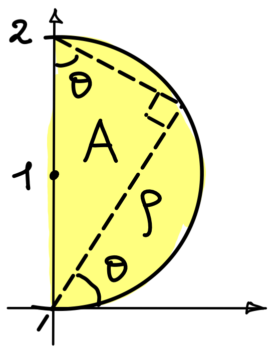
1) Parametrizzazione cartesiana di $S_1$:

$$\vec{\sigma}(x,y)=(x,y,\sqrt{4-x^2-y^2}) \text{ con } A=\{x^2+(y-1)^2\leq 1,\ x\geq 0\}.$$

Allora

$$\vec{\sigma}_x\times\vec{\sigma}_y=\left(\frac{x}{\sqrt{4-x^2-y^2}},\frac{y}{\sqrt{4-x^2-y^2}},1\right)$$

da cui $\|\vec{\sigma}_x\times\vec{\sigma}_y\|=\dfrac{2}{\sqrt{4-x^2-y^2}}$. Così




$$|S|=4\iint_A\|\vec{\sigma}_x\times\vec{\sigma}_y\|\,dx\,dy$$

$$\overset{CP}{=} 8\int_{\theta=0}^{\frac{\pi}{2}}\int_{\rho=0}^{2\sin\theta}\frac{1}{\sqrt{4-\rho^2}}\,\rho\,d\rho\,d\theta$$

$t=4-\rho^2$
$dt=-2\rho d\rho$

$$= 8\int_0^{\frac{\pi}{2}}\left(\int_{4\cos^2\theta}^{4}\frac{1}{\sqrt{t}}\cdot\frac{dt}{2}\right)d\theta=8\int_0^{\frac{\pi}{2}}\Big[\sqrt{t}\Big]_{4\cos^2\theta}^{4}d\theta$$

$$=8\int_0^{\frac{\pi}{2}}(2-2\cos\theta)\,d\theta=8\Big[2\theta-2\sin\theta\Big]_0^{\frac{\pi}{2}}=8(\pi-2).$$

2) Parametrizzazione in coordinate sferiche di $S_1$:

$$\vec{\sigma}(\theta,\varphi) = (2\cos\theta \operatorname{sen}\varphi, 2\operatorname{sen}\theta \operatorname{sen}\varphi, 2\cos\varphi)$$

con $A = \left\{(\theta,\varphi) : \theta \in \left[0, \frac{\pi}{2}\right], 0 \leq \varphi \leq \theta\right\}$ perchè la condizione $x^2+y^2 \leq 2y$ è equivalente a

$$(2\cos\theta \operatorname{sen}\varphi)^2 + (2\operatorname{sen}\theta \operatorname{sen}\varphi)^2 \leq 2 \cdot 2\operatorname{sen}\theta \operatorname{sen}\varphi$$

e, visto che nel primo ottante $\varphi \in \left[0, \frac{\pi}{2}\right]$, si ha

$$\cancel{4}\operatorname{sen}^2\varphi \leq \cancel{4}\operatorname{sen}\theta \,\cancel{\operatorname{sen}}\varphi \quad \text{ossia} \quad 0 \leq \varphi \leq \theta.$$

Ricordando che $\|\vec{\sigma}_\theta \times \vec{\sigma}_\varphi\| = R^2 \operatorname{sen}\varphi = 4\operatorname{sen}\varphi$ si ha

$$|S| = 4\iint_A \|\vec{\sigma}_\theta \times \vec{\sigma}_\varphi\| \, d\theta \, d\varphi = 4\int_{\theta=0}^{\frac{\pi}{2}} \left(\int_{\varphi=0}^{\theta} 4\operatorname{sen}\varphi \, d\varphi\right) d\theta$$

$$= 16\int_0^{\frac{\pi}{2}} \Big[-\cos\varphi\Big]_0^{\theta} d\theta = 16\int_0^{\frac{\pi}{2}} (1-\cos\theta)\, d\theta$$

$$= 16\Big[\theta - \operatorname{sen}\theta\Big]_0^{\frac{\pi}{2}} = 8(\pi - 2).$$

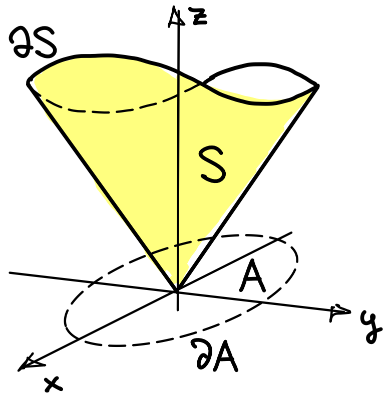
$\boxed{2.e}$ Calcolare l'area di

$$S=\{(x,y,z): z^2=x^2+y^2,\ z\geq 0,\ y^2+z^2\leq 1\}.$$

doppio cono con asse $z$ — cilindro pieno con asse $x$

Il bordo di $S$ è la curva data dall'intersezione del cono superiore e del cilindro

$$\begin{cases} z^2=x^2+y^2 \\ y^2+z^2=1 \\ z\geq 0 \end{cases}$$




Eliminando la variabile $z$ otteniamo la la proiezione di $\partial S$ sul piano $xy$

$$1-y^2=z^2=x^2+y^2 \iff x^2+2y^2=1 \iff x^2+\frac{y^2}{(1/\sqrt{2})^2}=1.$$

ellisse

Questo permette di scrivere una parametrizzazione cartesiana di $S$:

$$\vec{\sigma}(x,y)=(x,y,\sqrt{x^2+y^2}) \text{ con } A=\left\{(x,y): x^2+\frac{y^2}{(1/\sqrt{2})^2}\leq 1\right\}.$$

Quindi

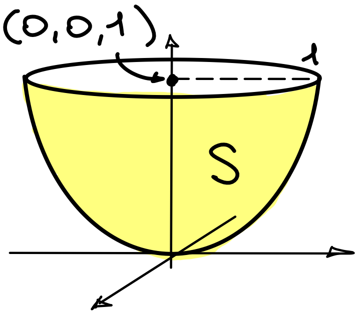
$$\|\vec{\sigma}_x\times\vec{\sigma}_y\|=\left\|\left(-\frac{x}{\sqrt{x^2+y^2}},-\frac{y}{\sqrt{x^2+y^2}},1\right)\right\|=\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}=\sqrt{2}.$$

Così

$$|S|=\iint_A \|\vec{\sigma}_x\times\vec{\sigma}_y\|\,dx\,dy=\sqrt{2}\,\underbrace{|A|}_{\text{Area ellisse}}=\sqrt{2}\cdot\pi\cdot 1\cdot\frac{1}{\sqrt{2}}=\pi.$$

$\boxed{2.7}$ Calcolare il baricentro

$$S=\{(x,y,z): z=x^2+y^2,\ z\in[0,1]\}.$$




Parametrizzazione cartesiana di S:

$$\vec{\sigma}(x,y)=(x,y,x^2+y^2) \quad \text{con} \quad A=\{(x,y): x^2+y^2\leq 1\}.$$

Quindi

$$\|\vec{\sigma}_x\times\vec{\sigma}_y\|=\|(-2x,-2y,1)\|=\sqrt{1+4(x^2+y^2)}.$$

Calcolo dell'area:

$$|S|=\iint_A \|\vec{\sigma}_x\times\vec{\sigma}_y\|\,dx\,dy \overset{CP}{=} \int_{\rho=0}^{1}\int_{\theta=0}^{2\pi}\sqrt{1+4\rho^2}\cdot\rho\,d\rho\,d\theta$$

$$\begin{cases} t=1+4\rho^2 \\ dt=8\rho\,d\rho \end{cases} \overset{\downarrow}{=} 2\pi\int_1^5\sqrt{t}\cdot\frac{dt}{8}=\frac{\pi}{4}\left[\frac{2t^{3/2}}{3}\right]_1^5=\frac{\pi}{6}\left(5^{3/2}-1\right).$$

Calcolo del baricentro: per simmetria $\bar{x}=\bar{y}=0$,

$$\bar{z}=\frac{1}{|S|}\iint_S z\,dS=\frac{1}{|S|}\iint_A (x^2+y^2)\|\vec{\sigma}_x\times\vec{\sigma}_y\|\,dx\,dy$$

$$\overset{CP}{=}\frac{1}{|S|}\int_{\rho=0}^{1}\int_{\theta=0}^{2\pi}\rho^2\sqrt{1+4\rho^2}\cdot\rho\,d\rho\,d\theta$$

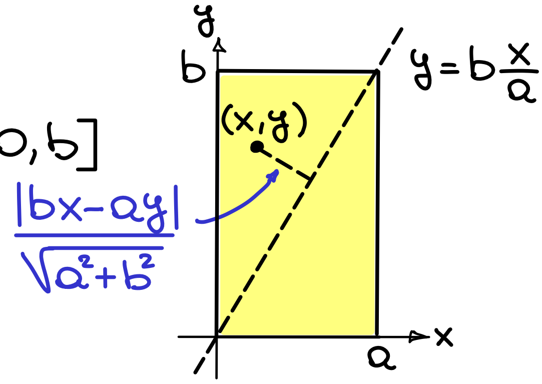
$$\begin{cases} t=1+4\rho^2 \\ dt=8\rho\,d\rho \end{cases} \overset{\downarrow}{=} \frac{2\pi}{|S|}\int_1^5\frac{t-1}{4}\sqrt{t}\,\frac{dt}{8}=\frac{\pi}{16|S|}\left[\frac{2t^{5/2}}{5}-\frac{2t^{3/2}}{3}\right]_1^5$$

$$=\frac{\pi}{16\cdot\frac{\pi}{6}\left(5^{3/2}-1\right)}\cdot\left(2\cdot5^{3/2}-\frac{2}{3}\cdot5^{3/2}-\frac{2}{5}+\frac{2}{3}\right)=\frac{1}{10}\cdot\frac{5^{5/2}+1}{5^{3/2}-1}.$$

**2.9** Calcolare $I/M$ del rettangolo

$S=\{(x,y,0): x\in[0,a], y\in[0,b]\}$ con $a,b>0$

rispetto all'asse $z$ e ad una sua diagonale.




Parametrizzazione di $S$:

$\vec{\sigma}(x,y)=(x,y,0)$ con $A=[0,a]\times[0,b]$

Quindi $\vec{\sigma}_x\times\vec{\sigma}_y=(0,0,1)$.

1) Asse $z$.

$$\frac{I}{M}=\frac{1}{|S|}\iint_S (x^2+y^2)\,dS=\frac{1}{ab}\iint_{[0,a]\times[0,b]}(x^2+y^2)\cdot 1\,dx\,dy$$

$$=\frac{1}{ab}\int_0^a\left[x^2y+\frac{y^3}{3}\right]_0^b dx=\frac{1}{ab}\int_0^a\left(bx^2+\frac{b^3}{3}\right)dx$$

$$=\frac{1}{ab}\left[b\frac{x^3}{3}+\frac{b^3x}{3}\right]_0^a=\frac{1}{ab}\cdot\frac{ba^3+b^3a}{3}=\frac{a^2+b^2}{3}.$$

2) Diagonale di $S$.

$$\frac{I}{M}=\frac{1}{|S|}\iint_S\frac{(bx-ay)^2}{a^2+b^2}\,dS=\frac{1}{ab(a^2+b^2)}\int_0^a\int_0^b(bx-ay)^2\,dx\,dy$$

$$=\frac{1}{ab(a^2+b^2)}\int_0^a\left[b^2x^2y-\cancel{2}abx\frac{y^2}{\cancel{2}}+a^2\frac{y^3}{3}\right]_0^b dx$$

$$=\frac{b^3}{ab(a^2+b^2)}\int_0^a\left(x^2-ax+\frac{a^2}{3}\right)dx$$

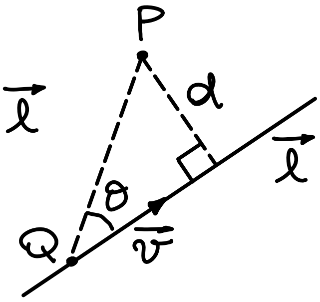
$$=\frac{b^3}{ab(a^2+b^2)}\left[\frac{x^3}{3}-a\frac{x^2}{2}+\frac{a^2}{3}x\right]_0^a=\frac{a^3b^3}{ab(a^2+b^2)}\cdot\frac{1}{6}=\frac{a^2b^2}{6(a^2+b^2)}.$$

OSSERVAZIONE

$d =$ distanza di $P$ da $\vec{\ell}$

$= \|P-Q\|\,|\operatorname{sen}\theta|$

$= \|(P-Q)\times\vec{v}\|$




retta $\vec{\ell}$ passante per $Q$ e direzione e versore $\vec{v}$

Nel caso di $Q=(0,0,0)$, $\vec{v}=\dfrac{(a,b,0)}{\sqrt{a^2+b^2}}$ e $P=(x,y,0)$.

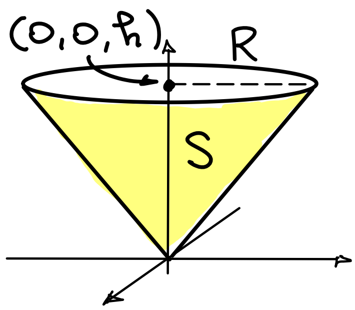
$$d=\left|\det\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ x & y & 0\\ \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} & 0\end{bmatrix}\right| = \frac{|bx-ay|}{\sqrt{a^2+b^2}}.$$

**2.h** Calcolare I/M di

$$S=\left\{(x,y,z): \frac{x^2+y^2}{R^2}=\frac{z^2}{h^2},\ z\in[0,h]\right\} \text{ con } h,R>0$$

rispetto agli assi $y$ e $z$.

S è la superficie laterale di un cono.




Parametrizzazione cartesiana di $S$:

$$\vec{\sigma}(x,y)=\left(x,y,\frac{h}{R}\sqrt{x^2+y^2}\right) \text{ con } A=\left\{(x,y): x^2+y^2\le R^2\right\}.$$

Allora

$$\|\vec{\sigma}_x\times\vec{\sigma}_y\|=\left\|\left(\frac{-hx}{R\sqrt{x^2+y^2}},\frac{-hy}{R\sqrt{x^2+y^2}},1\right)\right\|=\sqrt{1+\frac{h^2}{R^2}}=\frac{\sqrt{R^2+h^2}}{R}.$$

Calcolo dell'area di $S$:

$$|S|=\iint_A \|\vec{\sigma}_x\times\vec{\sigma}_y\|\,dx\,dy=\frac{\sqrt{R^2+h^2}}{R}|A|=\pi R\sqrt{R^2+h^2}$$

1) Asse y

$$\frac{I}{M}=\frac{1}{|S|}\iint_S (x^2+z^2)\,dS=\frac{1}{\pi R\cancel{\sqrt{R^2+h^2}}}\iint_A (x^2+\underbrace{z^2}_{\overset{\frac{h^2}{R^2}(x^2+y^2)}{=}})\cdot\frac{\cancel{\sqrt{R^2+h^2}}}{R}\,dx\,dy$$

$$\overset{CP}{=}\frac{1}{\pi R^2}\int_{\theta=0}^{2\pi}\left(\int_{\rho=0}^{R}\left(\rho^2\cos^2\theta+\frac{h^2}{R^2}\rho^2\right)\rho\,d\rho\right)d\theta$$

$$=\frac{1}{\pi R^2}\left(\int_0^{2\pi}\cos^2\theta\,d\theta+\frac{2\pi h^2}{R^2}\right)\cdot\left[\frac{\rho^4}{4}\right]_0^R$$

$$=\frac{1}{\pi R^2}\left(\pi+\frac{2\pi h^2}{R^2}\right)\frac{R^4}{4}=\frac{R^2}{4}+\frac{h^2}{2}.$$

2) Asse z

$$\frac{I}{M}=\frac{1}{|S|}\iint_S (x^2+y^2)\,dS=\frac{1}{\pi R\cancel{\sqrt{R^2+h^2}}}\iint_A (x^2+y^2)\cdot\frac{\cancel{\sqrt{R^2+h^2}}}{R}\,dx\,dy$$

$$\overset{CP}{=}\frac{1}{\pi R^2}\int_{\theta=0}^{2\pi}\left(\int_{\rho=0}^{R}\rho^2\cdot\rho\,d\rho\right)d\theta=\frac{\cancel{2\pi}}{\cancel{\pi}R^2}\cdot\left[\frac{\rho^4}{\cancel{4}_{\,2}}\right]_0^R=\frac{R^2}{2}.$$

$\boxed{2.i}$ Calcolare I/M di

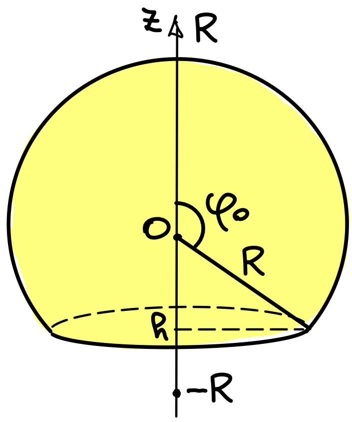
$$S=\{(x,y,z): x^2+y^2+z^2=R^2,\ z\geq h\}\ \text{con } h\in(-R,R) \text{ e } R>0$$

rispetto all'asse z.

Si ha che $h=R\cos\varphi_0$. Così

$$|S|=\int_{\theta=0}^{2\pi}\left(\int_{\varphi=0}^{\varphi_0}R^2\sin\varphi\,d\varphi\right)d\theta$$

$$=2\pi R^2\left[-\cos\varphi\right]_0^{\varphi_0}=2\pi R(R-h).$$



Quindi

$$\frac{I}{M} = \frac{1}{|S|} \iint_S (x^2+y^2)\, dS$$

$1 - \cos^2\varphi$

$$= \frac{1}{2\pi R(R-h)} \int_{\theta=0}^{2\pi} \left( \int_{\varphi=0}^{\varphi_0} R^2 \operatorname{sen}^2\varphi \cdot R^2 \operatorname{sen}\varphi \, d\varphi \right) d\theta$$

$(R-h)^2(2R+h)$

$$= \frac{R^3}{R-h} \left[ -\cos\varphi + \frac{\cos^3\varphi}{3} \right]_0^{\varphi_0} = \frac{1}{R-h} \cdot \frac{1}{3} \left( h^3 - 3R^2 h + 2R^3 \right)$$

$$= \frac{1}{3}(R-h)(2R+h)$$

**3.a** Verificare che se $g$ è $C^2$ allora $rot(\nabla g)=\vec{0}$.

$$rot(\nabla g) = det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_x & g_y & g_z \end{bmatrix}$$

Teorema di Schwarz

$$= (g_{zy}-g_{yz}, g_{xz}-g_{zx}, g_{yx}-g_{xy}) = (0,0,0).$$

Si noti che $rot(\nabla g)=\vec{0}$ è come dire che ogni campo conservativo $\nabla g$ è irrotazionale.

**3.b** Verificare che se $\vec{F}$ è $C^2$ allora

$$div(rot(\vec{F}))=0.$$

Si ha che

$$rot(\vec{F}) = det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix} = \left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

e quindi

Teorema di Schwarz

$$div(rot(\vec{F})) = \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0.$$

**3.c** Verificare che se $g$ e $\vec{F}$ sono $C^2$ allora

$$div(g\vec{F}) = g\, div(\vec{F}) + \langle \nabla g, \vec{F}\rangle.$$

Abbiamo che

$$div(g\vec{F}) = \frac{\partial}{\partial x}(gF_1) + \frac{\partial}{\partial y}(gF_2) + \frac{\partial}{\partial z}(gF_3)$$

$$= g_x F_1 + g\frac{\partial F_1}{\partial x} + g_y F_2 + g\frac{\partial F_2}{\partial y} + g_z F_3 + g\frac{\partial F_3}{\partial z}$$

$$= g\, div(\vec{F}) + \langle \nabla g, \vec{F}\rangle.$$

**3.d** Verificare che se $g$ e $\vec{F}$ sono $C^2$ allora
$$\mathrm{rot}(g\vec{F}) = g\,\mathrm{rot}(\vec{F}) + \nabla g \times \vec{F}.$$

Abbiamo che
$$\mathrm{rot}(g\vec{F}) = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ gF_1 & gF_2 & gF_3 \end{bmatrix}$$

$$= \left(\frac{\partial(gF_3)}{\partial y} - \frac{\partial(gF_2)}{\partial z}, \frac{\partial(gF_1)}{\partial z} - \frac{\partial(gF_3)}{\partial x}, \frac{\partial(gF_2)}{\partial x} - \frac{\partial(gF_1)}{\partial y}\right)$$

$$= \left(g_y F_3 + g\frac{\partial F_3}{\partial y} - g_z F_2 - g\frac{\partial F_2}{\partial z}, g_z F_1 + g\frac{\partial F_1}{\partial z} - g_x F_3 - g\frac{\partial F_3}{\partial x}, g_x F_2 + g\frac{\partial F_2}{\partial x} - g_y F_1 - g\frac{\partial F_1}{\partial y}\right)$$

$$= g\cdot\det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix} + \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ g_x & g_y & g_z \\ F_1 & F_2 & F_3 \end{bmatrix}$$

$$= \mathrm{rot}(g\vec{F}) = g\,\mathrm{rot}(\vec{F}) + \nabla g \times \vec{F}.$$

**3.e** Verificare che se $\vec{F} = (F_1, F_2, F_3)$ è $C^2$ allora

$$\mathrm{rot}(\mathrm{rot}(\vec{F})) = \nabla(\mathrm{div}(\vec{F})) - (\mathrm{div}(\nabla F_1), \mathrm{div}(\nabla F_2), \mathrm{div}(\nabla F_3)).$$

Verifichiamo l'uguaglianza per la prima componente (simile per le altre).

A sinistra si ha che
$$\mathrm{rot}(\vec{F}) = (F_{3y} - F_{2z}, F_{1z} - F_{3x}, F_{2x} - F_{1y})$$

e quindi la prima componente di $rot(rot(\vec{F}))$ è

$$\frac{\partial}{\partial y}(F_{2x} - F_{1y}) - \frac{\partial}{\partial z}(F_{1z} - F_{3x}) = F_{2xy} - F_{1yy} - F_{1zz} + F_{3xz} \quad (*)$$

A destra invece la prima componente di

$\nabla(div(\vec{F})) - (div(\nabla F_1), div(\nabla F_2), div(\nabla F_3))$ è

$$\frac{\partial}{\partial x}(F_{1x} + F_{2y} + F_{3z}) - div((F_{1x}, F_{1y}, F_{1z}))$$

$$= \cancel{F_{1xx}} + F_{2yx} + F_{3zx} - \cancel{F_{1xx}} - F_{1yy} - F_{1zz} \quad (**)$$

Infine, confrontando $(*)$ e $(**)$ si nota che sono uguali per il teorema di Schwarz.