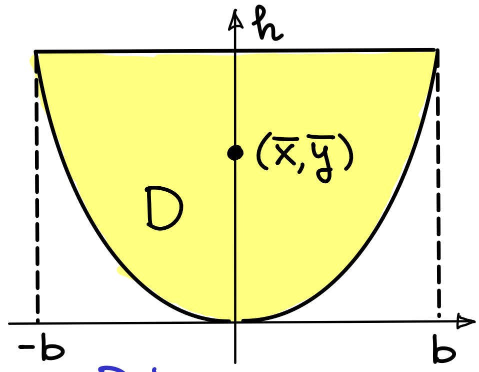
ANALISI MATEMATICA 2 - FOGLIO 6

$\boxed{1.a}$ Calcolare il baricentro di

$$D=\left\{(x,y) : \frac{hx^2}{b^2}\leq y\leq h\right\}$$

con $h,b>0$.



D è un segmento parabolico

Per simmetria

$$\overline{x}=\frac{1}{|D|}\iint_D x\,dxdy=0$$

Calcolo di $|D|$:

$$|D|=\iint_D 1\,dxdy=2\int_{x=0}^{b}\left(\int_{y=\frac{hx^2}{b^2}}^{h}dy\right)dx=2\int_0^b\left(h-\frac{hx^2}{b^2}\right)dx$$

$$=2h\left[x-\frac{x^3}{3b^2}\right]_0^b=2h\left(b-\frac{b}{3}\right)=\frac{4hb}{3}.$$

Calcolo di $\overline{y}$:

$$\overline{y}=\frac{1}{|D|}\iint_D y\,dxdy=\frac{2}{|D|}\int_{x=0}^{b}\left(\int_{y=\frac{hx^2}{b^2}}^{h}y\,dy\right)dx$$

$$=\frac{1}{|D|}\int_0^b\left[y^2\right]_{\frac{hx^2}{b^2}}^{h}dx=\frac{3}{4hb}\int_0^b\left(h^2-\frac{h^2x^4}{b^4}\right)dx$$

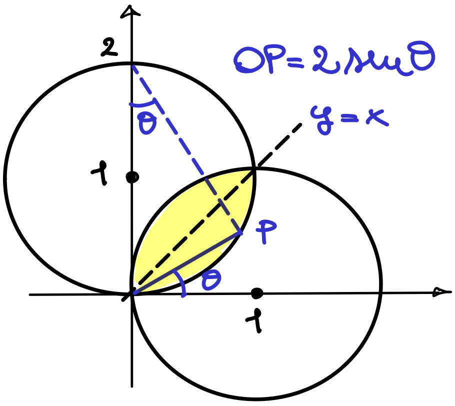
$$=\frac{3h^{\cancel{2}}}{4\cancel{h}b}\left[x-\frac{x^5}{5b^4}\right]_0^b=\frac{3h}{4\cancel{b}}\left(\cancel{b}-\frac{\cancel{b}}{5}\right)$$

$$=\frac{3h}{\cancel{4}}\cdot\frac{\cancel{4}}{5}=\frac{3h}{5}.$$

1.b

Calcolare $\iint_D \sqrt{x^2+y^2}\,dx\,dy$ con

$$D=\{(x,y): x^2+y^2 \le 2\min(x,y)\}.$$



$OP = 2\,\mathrm{sen}\,\theta$

$$\begin{cases} x^2+y^2 \le 2x \\ x^2+y^2 \le 2y \end{cases} \iff \begin{cases} (x-1)^2+y^2 \le 1 \\ x^2+(y-1)^2 \le 1 \end{cases}$$

x^2+y^2 e D sono simmetriche rispetto a $y=x$

$$2\int_0^{\pi/4}\left(\int_0^{2\,\mathrm{sen}\,\theta} \rho\cdot\rho\,d\rho\right)d\theta = 2\int_0^{\pi/4}\left[\frac{\rho^3}{3}\right]_0^{2\,\mathrm{sen}\,\theta} d\theta = \frac{16}{3}\int_0^{\pi/4} \underbrace{\mathrm{sen}^3\theta}_{(1-\cos^2\theta)\,\mathrm{sen}\,\theta}\,d\theta$$

$t=\cos\theta$, $dt=-\mathrm{sen}\,\theta$

$$= \frac{16}{3}\int_{1/\sqrt{2}}^{1}(1-t^2)\,dt = \frac{16}{3}\left[t-\frac{t^3}{3}\right]_{1/\sqrt{2}}^{1}$$

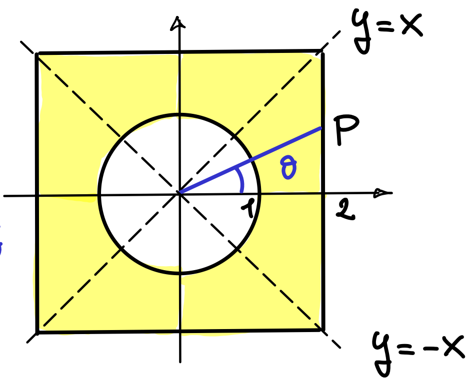
$$= \frac{16}{3}\left(\frac{2}{3}-\frac{1}{\sqrt{2}}\left(1-\frac{1}{6}\right)\right) = \frac{32}{9}-\sqrt{2}\,\frac{\cancel{16}^{\,4}}{3}\,\frac{5}{\cancel{12}_{\,3}} = \frac{32-20\sqrt{2}}{9}.$$

1.c

Calcolare $\iint_D \frac{dx\,dy}{(x^2+y^2)^{3/2}}$ con

$$D=\left\{(x,y): \begin{array}{c} 1\le x^2+y^2, \\ |x|\le 2,\ |y|\le 2 \end{array}\right\}$$

$OP = \dfrac{2}{\cos\theta}$



x^2+y^2 e D sono simmetrici rispetto a $x=0$, $y=0$ e $y=x$

$$=8\int_0^{\pi/4}\left(\int_1^{2/\cos\theta}\frac{1}{\rho^3}\,\rho\,d\rho\right)d\theta = 8\int_0^{\pi/4}\left[-\frac{1}{\rho}\right]_1^{2/\cos\theta}d\theta = 8\int_0^{\pi/4}\left(1-\frac{\cos\theta}{2}\right)d\theta$$

$$=8\left[\theta-\frac{\mathrm{sen}\,\theta}{2}\right]_0^{\pi/4} = \left(\frac{\pi}{4}-\frac{\sqrt{2}}{4}\right) = 2(\pi-\sqrt{2}).$$

1.d $\displaystyle \lim_{R\to+\infty} \iint_{D_R} \frac{1}{(1+x^2+y^2)^\alpha}\,dxdy$ per $\alpha>0$

dove $D_R=\{(x,y)\in\mathbb{R}^2:\ x^2+y^2\leq R^2\}$.

$$\iint_{D_R} \frac{1}{(1+x^2+y^2)^\alpha}\,dxdy \overset{CP}{=} \int_{\theta=0}^{2\pi}\int_{\rho=0}^{R} \frac{1}{(1+\rho^2)^\alpha}\rho\,d\rho\,d\theta$$

$t=1+\rho^2$, $dt=2\rho\,d\rho$

$$\overset{\downarrow}{=} \frac{\cancel{2}\pi}{\cancel{2}}\int_1^{1+R^2}\frac{1}{t^\alpha}\,dt = \begin{cases} \pi\left[\dfrac{t^{-\alpha+1}}{-\alpha+1}\right]_1^{1+R^2} & \text{se } \alpha\neq 1 \\ \pi\left[\log(t)\right]_1^{1+R^2} & \text{se } \alpha=1 \end{cases}$$

Quindi

$$\lim_{R\to+\infty}\iint_{D_R}\frac{1}{(1+x^2+y^2)^\alpha}\,dxdy = \begin{cases} \dfrac{\pi}{\alpha-1} & \text{se } \alpha>1 \\ +\infty & \text{se } \alpha\leq 1 \end{cases}$$

1.e $\displaystyle \lim_{r\to 0^+}\iint_{D_r}\frac{x^2\log(x^2+y^2)}{x^2+y^2}\,dxdy$

dove $D_r=\{(x,y)\in\mathbb{R}^2:\ r\leq x^2+y^2\leq 1,\ x\geq 0\}$.

$$\iint_{D_r}\frac{x^2\log(x^2+y^2)}{x^2+y^2}\,dxdy \overset{CP}{=} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\rho=r}^{1}\frac{\cancel{\rho^2}\cos^2\theta\log(\rho^2)}{\cancel{\rho^2}}\cdot\rho\,d\rho\,d\theta$$

$$=\underbrace{\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta\right)}_{=\frac{\pi}{2}}\int_r^1\log(\rho^2)\rho\,d\rho \overset{(*)}{=} \frac{\pi}{4}\left[\rho^2\log(\rho^2)-\rho^2\right]_r^1$$

$$=\frac{\pi}{4}\cdot\left(-1-(r^2\log(r^2)-r^2)\right)\xrightarrow{r\to 0^+} -\frac{\pi}{4}.$$

$(*)$ $\displaystyle\int\log(\rho^2)\rho\,d\rho=\frac{1}{2}\int\log(\rho^2)\,d\rho^2=\frac{1}{2}\left(\rho^2\log(\rho^2)-\rho^2\right)+c$

1.f

$$\lim_{r\to 0^+} \iint_{D_r} \frac{y^2}{x^2(1-y^2)}\,dx\,dy$$

dove $D_r=\{(x,y)\in\mathbb{R}^2 : r\leq y\leq 1-r,\ y\leq x\leq 1\}$.

$$\iint_{D_r} \frac{y^2}{x^2(1-y^2)}\,dx\,dy = \int_{y=r}^{1-r}\left(\int_{x=y}^{1}\frac{y^2}{x^2(1-y^2)}\,dx\right)dy$$

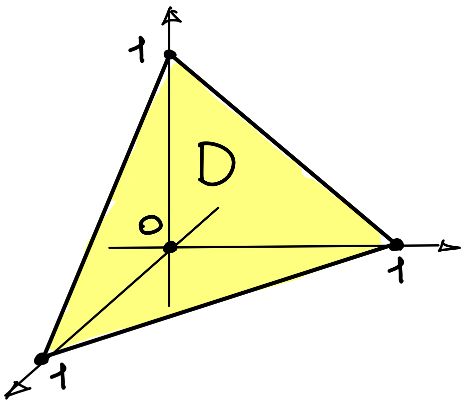
$$=\int_r^{1-r}\frac{y^2}{1-y^2}\overset{\frac{1}{y}-1}{\left[-\frac{1}{x}\right]_y^1}dy=\int_r^{1-r}\frac{y-y^2}{1-y^2}\,dy=\int_r^{1-r}\overset{1-\frac{1}{1+y}}{\frac{y}{1+y}}\,dy$$

$$=\Big[y-\log|1+y|\Big]_r^{1-r}=1-2r-\log\left(\frac{2-r}{1+r}\right)\xrightarrow{r\to0^+}1-\log 2.$$

1.g

$$\iiint_D \operatorname{sen}(\pi(x+y+z))\,dx\,dy\,dz$$

dove D è il tetraedro di vertici $(0,0,0),(1,0,0),(0,1,0),(0,0,1)$.



$D=\{(x,y,z) : x+y+z\leq 1,\ x,y,z\geq 0\}$

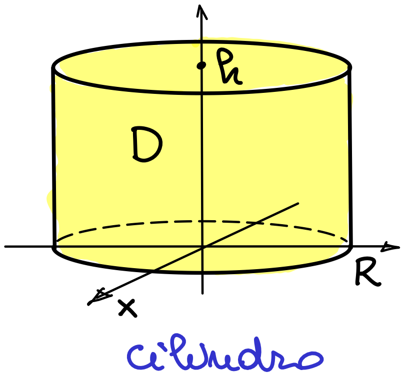
$$=\int_{x=0}^{1}\left(\int_{y=0}^{1-x}\left(\int_{z=0}^{1-x-y}\operatorname{sen}(\pi(x+y+z))\,dz\right)dy\right)dx$$

$$=\int_0^1\left(\int_0^{1-x}\left[-\frac{1}{\pi}\cos(\pi(x+y+z))\right]_0^{1-x-y}dy\right)dx=\frac{1}{\pi}\int_0^1\left(\int_0^{1-x}\left(1+\cos(\pi(x+y))\right)dy\right)dx$$

$$=\frac{1}{\pi}\cdot\frac{1}{2}+\frac{1}{\pi}\int_0^1\left[\frac{1}{\pi}\operatorname{sen}(\pi(x+y))\right]_0^{1-x}dx=\frac{1}{2\pi}+\frac{1}{\pi^2}\int_0^1\left(-\operatorname{sen}(\pi x)\right)dx$$

$$=\frac{1}{2\pi}+\frac{1}{\pi^2}\left[\frac{\cos(\pi x)}{\pi}\right]_0^1=\frac{1}{2\pi}-\frac{2}{\pi^3}.$$

$\boxed{1.h}$ Calcolare I/M per $D=\left\{(x,y,z):\begin{array}{l}x^2+y^2\le R^2\\ 0\le z\le h\end{array}\right\}$ con $h,R>0$ rispetto all'asse x.



La distanza di un punto (x,y,z) dall'asse x è $\sqrt{y^2+z^2}$.

Quindi

$$\frac{I}{M}=\frac{1}{|D|}\iiint_D (y^2+z^2)\,dx\,dy\,dz$$

$$\overset{cc}{=}\frac{1}{\pi R^2 h}\int_{\rho=0}^{R}\int_{\theta=0}^{2\pi}\int_{z=0}^{h}(\rho^2\sin^2\theta+z^2)\,\rho\,d\rho\,d\theta\,dz$$

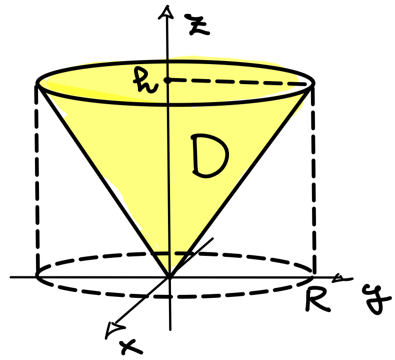
$$=\frac{1}{\pi R^2 h}\int_{\rho=0}^{R}\int_{\theta=0}^{2\pi}\left[\rho^3\sin^2\theta\cdot z+\rho\frac{z^3}{3}\right]_0^h d\rho\,d\theta$$

$$=\frac{\cancel{h}}{\pi R^2\cancel{h}}\int_{\rho=0}^{R}\rho^3 d\rho\cdot\int_{\theta=0}^{2\pi}\sin^2\theta\,d\theta+\frac{h^{\cancel{3}\,2}}{3\pi R^2\cancel{h}}\int_{\rho=0}^{R}\rho\,d\rho\cdot\int_{\theta=0}^{2\pi}d\theta$$

$$=\frac{1}{\cancel{\pi}\cancel{R^2}}\cdot\frac{R^{\cancel{4}\,2}}{4}\cdot\cancel{\pi}+\frac{h^2}{3\cancel{\pi}\cancel{R^2}}\cdot\frac{\cancel{R^2}}{\cancel{2}}\cdot\cancel{2}\cancel{\pi}$$

$$=\frac{R^2}{4}+\frac{h^2}{3}.$$

1.i Calcolare I/M per



$$D=\left\{(x,y,z): \frac{x^2+y^2}{R^2}\leq\frac{z^2}{h^2},\ 0\leq z\leq h\right\}$$

con $h, R>0$ rispetto agli assi y e z.

Ricordiamo che $|D|=\frac{\pi R^2\cdot h}{3}$.

1) Calcolo per l'asse y:

$$\frac{I}{M}=\frac{1}{|D|}\iiint_D (x^2+z^2)\,dx\,dy\,dz$$

$$\overset{CC}{=}\frac{3}{\pi R^2\cdot h}\int_{z=0}^{h}\left(\int_{\rho=0}^{R\frac{z}{h}}\left(\int_{\theta=0}^{2\pi}(\rho^2\cos^2\theta+z^2)\,d\theta\right)\rho\,d\rho\right)dz$$

$$=\frac{3}{\cancel{\pi}R^2\cdot h}\int_0^h\left(\int_0^{R\frac{z}{h}}\left(\rho^2\cdot\cancel{\pi}+z^2\cdot 2\cancel{\pi}\right)\rho\,d\rho\right)dz$$

$$=\frac{3}{R^2\cdot h}\int_0^h\left[\frac{\rho^4}{4}+z^2\rho^2\right]_0^{R\frac{z}{h}}dz=\frac{3}{\cancel{R^2}\cdot h}\int_0^h\left(\frac{R^{\cancel{4}\,2}z^4}{4h^4}+\frac{\cancel{R^2}z^4}{h^2}\right)dz$$

$$=\frac{3}{h}\left[\frac{R^2z^5}{20h^4}+\frac{z^5}{5h^2}\right]_0^h=\frac{3R^2}{20}+\frac{3h^2}{5}.$$

2) Calcolo per l'asse z:

$$\frac{I}{M}=\frac{1}{|D|}\iiint_D (x^2+y^2)\,dx\,dy\,dz$$

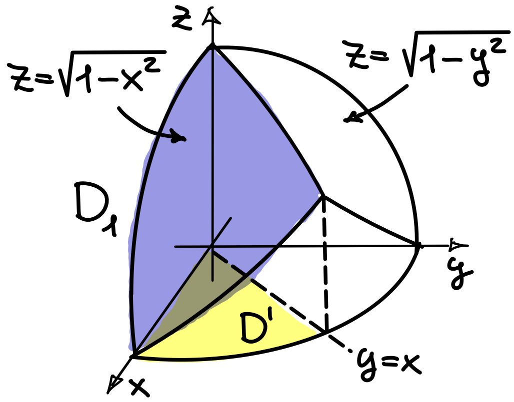
$$\overset{CC}{=}\frac{3}{\pi R^2\cdot h}\int_{z=0}^{h}\left(\int_{\rho=0}^{R\frac{z}{h}}\left(\int_{\theta=0}^{2\pi}\rho^2\,d\theta\right)\rho\,d\rho\right)dz=\frac{\cancel{6\pi}\ 3}{\cancel{\pi}R^2h}\int_0^h\left[\frac{\rho^4}{\cancel{4}\,2}\right]_0^{R\frac{z}{h}}dz$$

$$=\frac{3}{2\cancel{R^2}h}\int_0^h\frac{R^{\cancel{4}\,2}z^4}{h^4}\,dz=\frac{3R^2}{2h^5}\left[\frac{z^5}{5}\right]_0^h=\frac{3R^2}{10}.$$

$\boxed{1.j}$ Calcolare il volume di

$$D=\{(x,y,z):\ x^2+y^2\le 1,\ y^2+z^2\le 1,\ x^2+z^2\le 1\}.$$

$$(x,y,z)\in D \iff \begin{cases} x^2+y^2\le 1 \text{ e} \\ |z|\le \min\left(\sqrt{1-x^2},\sqrt{1-y^2}\right) \\ \qquad \sqrt{1-x^2} \text{ se } |y|\le|x| \end{cases}$$



Dato che D è simmetrico rispetto ai piani $x=0$, $y=0$, $z=0$ e $y=x$.

$|D|=8\cdot 2|D_1|$ dove

$$D_1=\{(x,y,z):(x,y)\in D',\ 0\le z\le\sqrt{1-x^2}\} \text{ e}$$

$$D'=\{(x,y):\ x\in[0,1],\ 0\le y\le x,\ 0\le y\le\sqrt{1-x^2}\}.$$

Calcolo di $|D_1|$:

$$|D_1|=\iint_{D'}\sqrt{1-x^2}\,dx\,dy \overset{CP}{=} \int_{\theta=0}^{\pi/4}\int_{\rho=0}^{1}\sqrt{1-\rho^2\cos^2\theta}\,\rho\,d\rho\,d\theta$$

$u=1-\rho^2\cos^2\theta$
$du=-2\rho\cos^2\theta\,d\rho$

$$=\int_0^{\pi/4}\frac{1}{2\cos^2\theta}\left(\int_{\operatorname{sen}^2\theta}^{1}\sqrt{u}\,du\right)d\theta=\int_0^{\pi/4}\frac{1}{\cancel{2}\cos^2\theta}\left[\frac{\cancel{2}u^{3/2}}{3}\right]_{\operatorname{sen}^2\theta}^{1}d\theta$$

$$=\frac{1}{3}\int_0^{\pi/4}\frac{1-\operatorname{sen}^3\theta}{\cos^2\theta}\,d\theta=\frac{1}{3}\int_0^{\pi/4}\frac{1}{\cos^2\theta}\,d\theta+\frac{1}{3}\int_0^{\pi/4}\frac{1-\cos^2\theta}{\cos^2\theta}\,d(\cos\theta)$$

$$=\frac{1}{3}\left[\operatorname{tg}\theta\right]_0^{\pi/4}+\frac{1}{3}\left[-\frac{1}{\cos\theta}-\cos\theta\right]_0^{\pi/4}$$

$$=\frac{1}{3}+\frac{1}{3}\left(-\sqrt{2}-\frac{\sqrt{2}}{2}+2\right)=1-\frac{\sqrt{2}}{2}.$$

Infine

$$|D|=16|D_1|=8(2-\sqrt{2}).$$

2.a Calcolare $\int_\gamma f\,ds$ dove $f(x,y)=x^2+y^2$

e $\overline{\gamma}(t)=(e^{-t}\operatorname{sen}t,e^{-t}\cos t)$ per $t\in[0,2\pi]$.

Si ha che

$$\overline{\gamma}'(t)=(-e^{-t}\operatorname{sen}t+e^{-t}\cos t,-e^{-t}\cos t-e^{-t}\operatorname{sen}t)$$

e quindi

$$\|\overline{\gamma}'(t)\|=e^{-t}\Big(\underbrace{(-\operatorname{sen}t+\cos t)^2+(-\cos t-\operatorname{sen}t)^2}_{2(\operatorname{sen}^2t+\cos^2t)+0}\Big)^{1/2}=\sqrt{2}\,e^{-t}.$$

Infine

$$\int_\gamma f\,ds=\int_0^{2\pi}e^{-2t}(\operatorname{sen}^2t+\cos^2t)\cdot\sqrt{2}\,e^{-t}\,dt$$

$$=\sqrt{2}\left[-\frac{e^{-3t}}{3}\right]_0^{2\pi}=\frac{\sqrt{2}}{3}(1-e^{-6\pi}).$$

2.b Calcolare $\int_\gamma f\,ds$ dove $f(x,y)=xy$ e

$$\gamma=\{(x,y):4x^2+y^2=4,\ x\geq 0,\ y\geq 0\}.$$

$\hookrightarrow x^2+\left(\frac{y}{2}\right)^2=1$ arco di ellisse nel 1° quadrante

Parametrizzazione di γ:

$$\overline{\gamma}(t)=(\cos t,2\operatorname{sen}t)\quad\text{con } t\in\left[0,\tfrac{\pi}{2}\right]$$

allora

$$\overline{\gamma}'(t)=(-\operatorname{sen}t,2\cos t)\Rightarrow\|\overline{\gamma}'(t)\|=\sqrt{1+3\cos^2t}.$$

Quindi

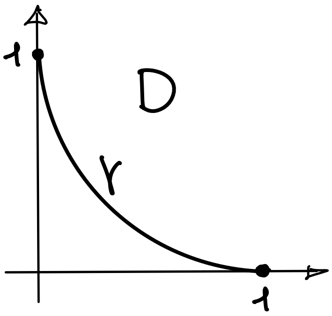
$$\int_\gamma f\,ds=\int_0^{\pi/2}\cos t\cdot 2\operatorname{sen}t\sqrt{1+3\cos^2t}\,dt$$

$u=1+3\cos^2t$

$du=6\cos t\cdot(-\operatorname{sen}t)\,dt$

$$=\int_4^1\sqrt{u}\left(-\frac{du}{3}\right)=\frac{1}{3}\left[\frac{2u^{3/2}}{3}\right]_1^4=\frac{2}{9}(8-1)=\frac{14}{9}.$$

2.C Calcolare la lunghezza di
$$\gamma=\{(x,y): x^{2/3}+y^{2/3}=1,\ x,y\geq 0\}.$$

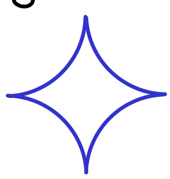


Esplicitando la y si ottiene
$$y(x)=(1-x^{2/3})^{3/2} \text{ con } x\in[0,1].$$
Quindi una parametrizzazione di γ è
$$\vec{\gamma}(t)=(t,(1-t^{2/3})^{3/2}) \text{ con } t\in[0,1].$$
Allora per $t\in[0,1]$:
$$\|\vec{\gamma}\,'(t)\|^2=1^2+\left(\frac{\cancel{3}}{\cancel{2}}(1-t^{2/3})^{1/2}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot t^{-\frac{1}{3}}\right)^2$$
$$=1+(1-t^{2/3})\cdot t^{-2/3}=t^{-2/3}.$$ non definito per $t=0$

Così
$$|\gamma|=\int_0^1\|\vec{\gamma}\,'(t)\|dt=\int_{0^+}^1 t^{-1/3}dt=\left[\frac{3t^{2/3}}{2}\right]_0^1=\frac{3}{2}.$$

Svolgimento alternativo.
Se $t=\cos^3(s)$, un'altra parametrizzazione di γ è
$$\vec{\gamma}_2(s)=(\cos^3(s),(1-\cos^2(s))^{3/2})=(\cos^3(s),\operatorname{sen}^3(s))$$
con $s\in[0,\frac{\pi}{2}]$. In tal caso
$$\|\vec{\gamma}_2\,'(s)\|^2=(3\cos^2(s)\cdot(-\operatorname{sen}(s)))^2+(3\operatorname{sen}^2(s)\cdot\cos(s))^2$$
$$=9\cos^4(s)\cdot\operatorname{sen}^2(s)+9\operatorname{sen}^4(s)\cos^2(s)$$
$$=9\operatorname{sen}^2(s)\cos^2(s)=\frac{9}{4}\operatorname{sen}^2(2s).$$
Così
$$|\gamma|=\int_0^{\pi/2}\|\vec{\gamma}_2\,'(s)\|ds=\frac{3}{2}\int_0^{\pi/2}\underbrace{\operatorname{sen}(2s)}_{\geq 0}ds=\frac{3}{4}\Big[-\cos(2s)\Big]_0^{\pi/2}=\frac{3}{2}.$$

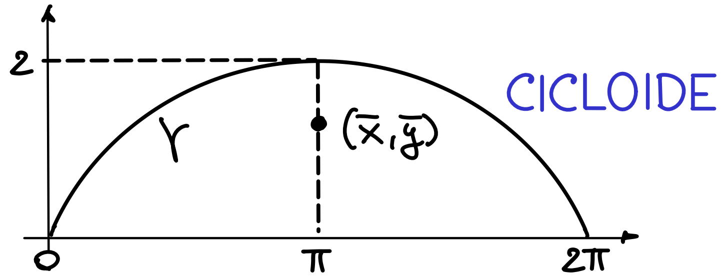
OSSERVAZIONE Per simmetria la lunghezza di
$$\gamma=\{(x,y): x^{2/3}+y^{2/3}=1\} \text{ è } 4\cdot\frac{3}{2}=6$$


ASTROIDE

$\boxed{2.d}$ Calcolare il baricentro della curva

$$\vec{\gamma}(t)=(t-\operatorname{sen}t,\,1-\cos t) \quad \text{con } t\in[0,2\pi]$$

$$\vec{\gamma}(0)=(0,0)$$
$$\vec{\gamma}(\pi)=(\pi,2)$$
$$\vec{\gamma}(2\pi)=(2\pi,0)$$



Calcolo della lunghezza $|\gamma|$.

$$\vec{\gamma}'(t)=(1-\cos t,\,\operatorname{sen}t)$$

$$\|\vec{\gamma}'(t)\|^2=(1-\cos t)^2+\operatorname{sen}^2 t=2-2\cos t=4\operatorname{sen}^2\left(\tfrac{t}{2}\right)$$

$$|\gamma|=\int_0^{2\pi}2\left|\operatorname{sen}\left(\tfrac{t}{2}\right)\right|dt \overset{u=\frac{t}{2}}{=} 4\int_0^{\pi}\operatorname{sen}(u)\,du=4\left[-\cos(u)\right]_0^{\pi}=8.$$

Quindi

$$\bar{x}=\frac{1}{|\gamma|}\int_\gamma x\,ds=\frac{1}{\cancel{8}\,4}\int_0^{2\pi}(t-\operatorname{sen}t)\cdot\cancel{2}\operatorname{sen}\left(\tfrac{t}{2}\right)dt$$

$$\overset{u=t-\pi}{=}\frac{1}{4}\int_{-\pi}^{\pi}\Big(u+\pi-\underbrace{\operatorname{sen}(u+\pi)}_{=-\operatorname{sen}u}\Big)\underbrace{\operatorname{sen}\left(\tfrac{u}{2}+\tfrac{\pi}{2}\right)}_{=\cos\left(\frac{u}{2}\right)}du$$

$$=\frac{1}{4}\int_{-\pi}^{\pi}\Big(\underbrace{u\cos\left(\tfrac{u}{2}\right)}_{\text{disp}}+\underbrace{\pi\cos\left(\tfrac{u}{2}\right)}_{\text{pari}}+\underbrace{\operatorname{sen}u\cos\left(\tfrac{u}{2}\right)}_{\text{disp}}\Big)du$$

$$=\frac{\cancel{2}\pi}{\cancel{4}\,2}\int_0^{\pi}\cos\left(\tfrac{u}{2}\right)du=\frac{\pi}{2}\left[2\operatorname{sen}\left(\tfrac{u}{2}\right)\right]_0^{\pi}=\pi.$$

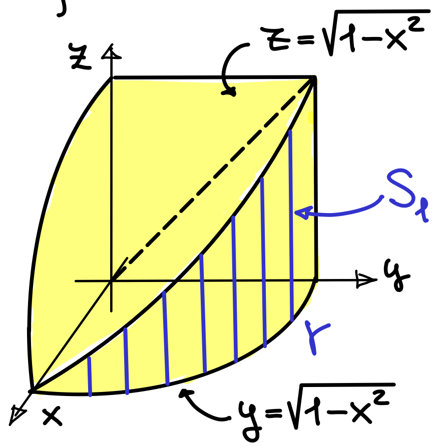
Inoltre

$$\bar{y}=\frac{1}{|\gamma|}\int_\gamma y\,ds=\frac{1}{\cancel{8}\,2}\int_0^{2\pi}\underbrace{(1-\cos t)}_{=\cancel{2}\operatorname{sen}^2\left(\frac{t}{2}\right)}\cdot\cancel{2}\operatorname{sen}\left(\tfrac{t}{2}\right)dt$$

$$\overset{u=\frac{t}{2}}{=}\int_0^{\pi}(1-\cos^2 u)\,d(-\cos u)=\left[-\cos u+\frac{\cos^3 u}{3}\right]_0^{\pi}=2\left(1-\frac{1}{3}\right)=\frac{4}{3}.$$

2.e Calcolare l'area della superficie di
$$D=\{(x,y,z): x^2+y^2\leq 1,\ x^2+z^2\leq 1\}.$$

La superficie S del solido D è composta da 8·2 parti che, per le simmetrie rispetto ai piani $x=0, y=0, z=0, y=z$, hanno la stessa area. Una di queste parti è



$$S_1=\{(x,y,z): y=\sqrt{1-x^2},\ 0\leq z\leq\sqrt{1-x^2},\ x\geq 0, y\geq 0\}.$$

Quindi $|S_1|=\int_\gamma f\,ds=\int_0^{\pi/2} f(\bar\gamma(t))\cdot\|\bar\gamma'(t)\|\,dt$

con $\bar\gamma(t)=(\cos t, \operatorname{sen} t)$ per $t\in[0,\frac{\pi}{2}]$ e $f(x,y)=\sqrt{1-x^2}$.

Così $\|\bar\gamma'(t)\|^2=\|(-\operatorname{sen} t,\cos t)\|^2=(-\operatorname{sen} t)^2+(\cos t)^2=1$,

$$|S_1|=\int_0^{\pi/2}\underbrace{\sqrt{1-\cos^2 t}}_{|\operatorname{sen} t|}\cdot 1\,dt=\left[-\cos t\right]_0^{\frac{\pi}{2}}=1$$

e

$$|S|=16|S_1|=16.$$

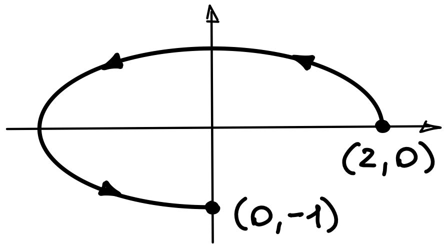
3.a Calcolare $\int_\gamma \langle \vec{F}, d\vec{s}\rangle$ dove $\vec{F}(x,y)=(x+y,\,x-y)$

e $\vec{\gamma}(t)=(2\cos t,\sin t)$ per $t\in[0,\frac{3\pi}{2}]$.

$\vec{F}$ è $C^1(\mathbb{R}^2)$ ed è irrotazionale $\frac{\partial F_1}{\partial y}=1=\frac{\partial F_2}{\partial x}$.

Quindi $\vec{F}$ è conservativo in $\mathbb{R}^2$ (semplicemente connesso)

Potenziale: dobbiamo risolvere



$$\frac{\partial U}{\partial x}=x+y \quad \text{e} \quad \frac{\partial U}{\partial y}=x-y.$$

Allora

$$U(x,y)=\int (x+y)\,dx=\frac{x^2}{2}+xy+c(y) \quad \text{e}$$

$$\frac{\partial U}{\partial y}=\frac{\partial}{\partial y}\left(\frac{x^2}{2}+xy+c(y)\right)=0+x+c'(y)\overset{?}{=}x-y \Rightarrow c'(y)=-y$$

$$\Rightarrow c(y)=-\frac{y^2}{2}+c.$$

per il calcolo poniamo $c=0$

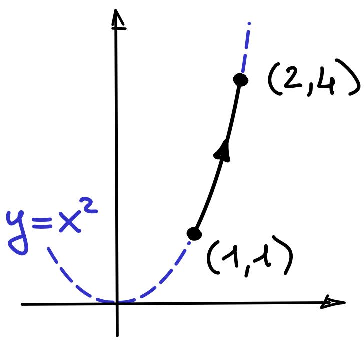
Così $U(x,y)=\frac{x^2}{2}+xy-\frac{y^2}{2}+\cancel{c}$ e

$$\int_\gamma \langle \vec{F}, d\vec{s}\rangle = U(\underbrace{\vec{\gamma}(\tfrac{3\pi}{2})}_{(0,-1)})-U(\underbrace{\vec{\gamma}(0)}_{(2,0)})=-\frac{1}{2}-2=-\frac{5}{2}.$$

3.b Calcolare $\int_\gamma \langle \vec{F}, d\vec{s}\rangle$ dove $\vec{F}(x,y)=\left(\frac{1}{x+y},\,x\right)$

e $\vec{\gamma}(t)=(t,t^2)$ per $t\in[1,2]$.

Dato che $\vec{\gamma}'(t)=(1,2t)$, si ha



$$\int_\gamma \langle \vec{F}, d\vec{s}\rangle=\int_1^2 \left(\overbrace{\frac{1}{t+t^2}}^{\frac{1}{t}-\frac{1}{t+1}}+2t\cdot t\right)dt$$

$$=\left[\log\left(\frac{t}{t+1}\right)+\frac{2t^3}{3}\right]_1^2$$

$$=\log\left(\frac{2}{3}\right)+\frac{16}{3}-\log\left(\frac{1}{2}\right)-\frac{2}{3}=\log\left(\frac{4}{3}\right)+\frac{14}{3}.$$

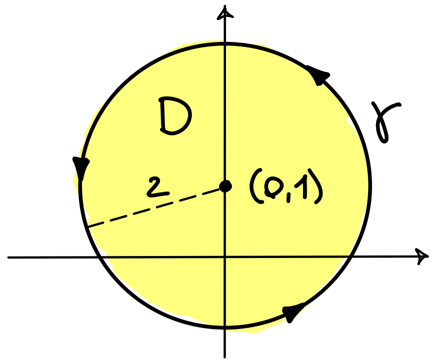
3.C Calcolare $\int_\gamma \langle \vec{F}, d\vec{s} \rangle$ dove

$$\vec{F}(x,y) = \left(y + \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right)$$

e γ è il bordo di

$$D = \{(x,y) : x^2 + (y-1)^2 \leq 4\}$$

percorso in senso antiorario.



Osserviamo che $\vec{F} = \vec{F}_1 + \vec{F}_2$ dove

$$\vec{F}_1(x,y) = (y, 0) \quad e \quad \vec{F}_2(x,y) = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right)$$

$\vec{F}_2$ è conservativo in $\mathbb{R}^2 \setminus \{(0,0)\}$ con potenziale $U(x,y) = \log(x^2+y^2)$. Dato che γ è un percorso chiuso si ha che $\int_\gamma \langle \vec{F}_2, d\vec{s} \rangle = 0$.

$\vec{F}_1$ non è irrotazionale: $\frac{\partial(y)}{\partial y} = 1 \neq 0 = \frac{\partial(0)}{\partial x}$.

Quindi

$$\begin{aligned}
\int_\gamma \langle \vec{F}, d\vec{s} \rangle &= \int_\gamma \langle \vec{F}_1, d\vec{s} \rangle + \int_\gamma \langle \vec{F}_2, d\vec{s} \rangle \\
&= \int_0^{2\pi} \langle \vec{F}_1(\vec{\gamma}(t)), \vec{\gamma}'(t) \rangle \, dt + 0 \\
&= \int_0^{2\pi} \left((1 + 2\sin t) \cdot (-2\sin t) + 0 \cdot (2\cos t) \right) dt \\
&= -2 \int_0^{2\pi} \sin t \, dt - 4 \int_0^{2\pi} \sin^2 t \, dt = 0 - 4\pi = -4\pi
\end{aligned}$$

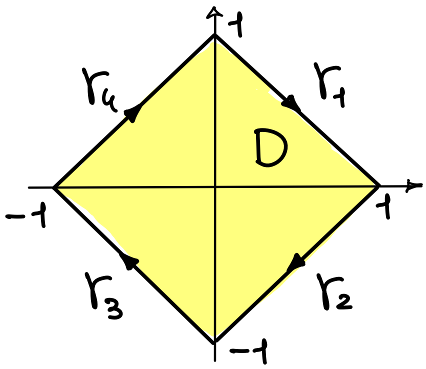
dove $\vec{\gamma}(t) = (2\cos t, 1 + 2\sin t)$ per $t \in [0, 2\pi]$.

$\boxed{3.d}$ Calcolare $\int_\gamma \langle \vec{F}, d\vec{s} \rangle$ dove $\vec{F}(x,y) = \left(\frac{x+y}{x+y+2}, -\frac{1}{x+y+2} \right)$

e γ è il bordo di

$$D = \{ (x,y) : |x| + |y| \leq 1 \}$$

percorso in senso orario.



Notiamo che in D $x+y+2 \neq 0$

e quindi $\vec{F}$ è continuo in D

Parametrizzazione di ∂D: per $t \in [0,1]$

$$\vec{\gamma}_1(t) = (t, 1-t) \qquad \vec{\gamma}_2(t) = (1-t, -t)$$

$$\vec{\gamma}_3(t) = (-t, -1+t) \qquad \vec{\gamma}_4(t) = (-1+t, t)$$

Ora si può fare il calcolo diretto con $\vec{F}$ oppure osservare che $\vec{F} = \vec{F}_1 + \vec{F}_2$,

$$\vec{F}_1(x,y) = \left(-\frac{1}{x+y+2}, 0 \right), \quad \vec{F}_2(x,y) = \left(1 - \frac{1}{x+y+2}, -\frac{1}{x+y+2} \right)$$

dove $\vec{F}_2$ è conservativo in D con potenziale

$U(x,y) = x - \log(|x+y+2|)$. Allora

$$\int_\gamma \langle \vec{F}, d\vec{s} \rangle = \int_\gamma \langle \vec{F}_1, d\vec{s} \rangle = \sum_{i=1}^{4} \int_{\gamma_i} \langle \vec{F}_1, d\vec{s} \rangle$$

$$= \int_0^1 \frac{-1}{3}(+1)\,dt + \int_0^1 \frac{-1}{3-2t}(-1)\,dt + \int_0^1 \frac{-1}{1}(-1)\,dt + \int_0^1 \frac{-1}{2t+1}(+1)\,dt$$

$$= -\frac{1}{3} + \left[-\frac{1}{2}\log|3-2t| \right]_0^1 + 1 - \left[\frac{1}{2}\log|2t+1| \right]_0^1$$

$$= \frac{2}{3} + \cancel{\frac{1}{2}\log 3} - \cancel{\frac{1}{2}\log 3} = \frac{2}{3}.$$

$\boxed{3.e}$ Calcolare $\int_\gamma \langle \vec{F}, d\vec{s} \rangle$ dove

$\vec{F}(x,y,z) = (2yz + 3x^2, 2xz + e^y, 2xy + x^2 z)$ e

$\vec{\gamma}(t) = (\cos t, \sin t, t)$ con $t \in [0, 3\pi]$.

Consideriamo $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$:

1) $\vec{F}_1(x,y,z) = (3x^2, e^y, 0)$ è conservativo in $\mathbb{R}^3$ con potenziale

$$U_1(x,y,z) = \int 3x^2 dx + \int e^y dy = x^3 + e^y + \cancel{c}$$

2) $\vec{F}_2(x,y,z) = (2yz, 2xz, 2xy)$ è irrotazionale in $\mathbb{R}^3$ e quindi è conservativo con potenziale

$$U_2(x,y,z) = \int 2yz\, dx = 2xyz + c(y,z)$$

e

$$\frac{\partial U_2}{\partial y} = 2xz + \frac{\partial c}{\partial y} = 2xz \,, \quad \frac{\partial U_2}{\partial z} = 2xy + \frac{\partial c}{\partial z} = 2xy$$

implicano che $\nabla c = (0,0)$ ossia c è costante.

Quindi $U_2(x,y,z) = 2xyz + \cancel{c}$.

3) $\vec{F}_3(x,y,z) = (0,0,x^2 z)$ non è irrotazionale in $\mathbb{R}^3$

$$0 = \frac{\partial (0)}{\partial z} \neq \frac{\partial (x^2 z)}{\partial x} = 2xz \quad \text{per } x \neq 0, z \neq 0.$$

Inoltre $\vec{\gamma}'(t) = (-\sin t, \cos t, 1)$ per $t \in [0, 3\pi]$.

Così

$$\int_\gamma \langle \vec{F}_3, d\vec{s} \rangle = \int_0^{3\pi} \langle \vec{F}_3(\vec{\gamma}(t)), \vec{\gamma}'(t) \rangle dt$$

$$= \int_0^{3\pi} \underbrace{\cos^2 t}_{\frac{\cos(2t)+1}{2}} \cdot t \cdot 1\, dt = \frac{1}{2}\left[\frac{t^2}{2}\right]_0^{3\pi} + \frac{1}{2}\int_0^{3\pi} \underbrace{\cos(2t) \cdot t\, dt}_{\frac{t}{2} d(\sin(2t))}$$

$$= \frac{9\pi^2}{4} + \frac{1}{4}\Big[t \cdot \operatorname{sen}(2t)\Big]_0^{3\pi} - \frac{1}{4}\int_0^{3\pi} \operatorname{sen}(2t)\,dt$$

$$= \frac{9\pi^2}{4} + 0 + \frac{1}{8}\Big[\cos(2t)\Big]_0^{3\pi} = \frac{9\pi^2}{4}.$$

Infine $\vec{\gamma}(0) = (1,0,0)$, $\vec{\gamma}(3\pi) = (-1,0,3\pi)$ e

$$\int_\gamma \langle \vec{F}, d\vec{s} \rangle = U_1(-1,0,3\pi) - U_1(1,0,0)$$

$$+ U_2(-1,0,3\pi) - U_2(1,0,0) + \int_\gamma \langle \vec{F}_3, d\vec{s} \rangle$$

$$= (-1)^3 + e^0 - (1^3 + e^0) + 0 - 0 + \frac{9\pi^2}{4}$$

$$= -2 + \frac{9\pi^2}{4}.$$