ANALISI MATEMATICA 2 - FOGLIO 5

1.a $f(x,y,z)=x+z$ Max/min in
$$D=\left\{(x,y,z):\ \begin{matrix}x^2+y^2+z^2=2\\ z=x^2+y^2\end{matrix}\right\}$$

$f\in C^2(\mathbb{R}^3)$, D è chiuso e limitato. Poniamo
$$g_1(x,y,z)=x^2+y^2+z^2-2=0,\quad g_2(x,y,z)=x^2+y^2-z=0.$$

Regolarità:
$$\begin{bmatrix} g_{1x} & g_{1y} & g_{1z}\\ g_{2x} & g_{2y} & g_{2z}\end{bmatrix}=\begin{bmatrix} 2x & 2y & 2z\\ 2x & 2y & -1\end{bmatrix}$$

Il rango non vale 2 se e solo se
$$\begin{cases}\det\begin{bmatrix}2x & 2z\\ 2x & -1\end{bmatrix}=2x(-1-2z)=0\\ \det\begin{bmatrix}2y & 2z\\ 2y & -1\end{bmatrix}=2y(-1-2z)=0\end{cases}\iff \begin{matrix}(0,0,z)\notin D\\ (x,y,-\frac{1}{2})\notin D\end{matrix}$$

Tutti i punti di D sono regolari.

Moltiplicatori:
$$\begin{cases}1=\lambda 2x+\mu 2x & \to\ 1=2x(\lambda+\mu)\to\lambda+\mu\neq 0\\ 0=\lambda 2y+\mu 2y & \to\ 2y\underbrace{(\lambda+\mu)}_{\neq 0}=0\to y=0\\ 1=\lambda 2z-\mu\\ x^2+y^2+z^2=2\\ z=x^2+y^2\end{cases}$$

$$\begin{cases}y=0\\ 1=2(\lambda+\mu)x\\ 1=2\lambda z-\mu\\ x^2+z^2=2\ \to\ z+z^2=2\\ z=x^2\end{cases}$$

$z+z^2=2$: $z=1 \to x=1,\ x=-1$; $z=-2=x^2 \to \emptyset$

Punti stazionari vincolati:

$(1,0,1)$ $\lambda=\frac{1}{2},\mu=0$

$(-1,0,1)$ $\lambda=\frac{1}{6},\mu=-\frac{2}{3}$

$$\boxed{\begin{matrix}f(1,0,1)=2\\ f(-1,0,1)=0\end{matrix}}\ \begin{matrix}\text{Max}\\ \text{Min}\end{matrix}$$

$(1,0,1)$ punto di max assoluto

$(-1,0,1)$ punto di min assoluto

1.b $f(x,y,z)=2x+2y-z$ Max/min in

$$D=\{(x,y,z): x^2+y^2+z^2\leq 4\}$$

$f\in C^2(\mathbb{R}^3)$, D è chiuso e limitato.

Dato che

$$\nabla f(x,y,z)=(2,2,-1)\neq(0,0,0)$$

non ci sono punti stazionari interni a D.

Bordo: $\partial D=\{(x,y,z): x^2+y^2+z^2=4\}$

Sia $g(x,y,z)=x^2+y^2+z^2-4=0$. Allora

$$\nabla g(x,y,z)=(2x,2y,2z)=(0,0,0)$$

se e solo se $(x,y,z)=(0,0,0)\notin\partial D$.

I punti di D sono tutti regolari.

Moltiplicatori:

$$\begin{cases} 2=\lambda 2x \xrightarrow{\lambda\neq 0} x=1/\lambda \\ 2=\lambda 2y \longrightarrow y=1/\lambda \\ -1=\lambda 2z \longrightarrow z=-1/2\lambda \\ x^2+y^2+z^2=4 \longrightarrow \frac{1}{\lambda^2}+\frac{1}{\lambda^2}+\frac{1}{4\lambda^2}=4 \longrightarrow \lambda^2=\frac{9}{16} \end{cases} \quad \begin{matrix}\nearrow \lambda=\frac{3}{4} \\ \searrow \lambda=-\frac{3}{4}\end{matrix}$$

Punti stazionari vincolati:

$$\left(\frac{4}{3},\frac{4}{3},-\frac{2}{3}\right) \quad \lambda=\frac{3}{4} \qquad f\left(\frac{4}{3},\frac{4}{3},-\frac{2}{3}\right)=6 \quad \text{Max}$$

$$\left(-\frac{4}{3},-\frac{4}{3},\frac{2}{3}\right) \quad \lambda=-\frac{3}{4} \qquad f\left(-\frac{4}{3},-\frac{4}{3},\frac{2}{3}\right)=-6 \quad \text{Min}$$

$\left(\frac{4}{3},\frac{4}{3},-\frac{2}{3}\right)$ punto di max. assoluto

$\left(-\frac{4}{3},-\frac{4}{3},\frac{2}{3}\right)$ punto di min. assoluto

$\boxed{1.c}$ $f(x,y,z)=4x+z^2$ Max/min in

$$D=\{(x,y,z): x^2+y^2\le z\le 4\}$$

$f\in C^2(\mathbb{R}^3)$, D è chiuso e limitato.

Dato che $\nabla f(x,y,z)=(4,0,2z)\neq(0,0,0)$ non ci sono punti stazionari interni a D.

Bordo: $\partial D=\Gamma_1\cup\Gamma_2$ dove

$\Gamma_1=\{(x,y,z): z=4, x^2+y^2\le 4\}$ e $\Gamma_2=\{(x,y,z): x^2+y^2=z, z<4\}$

1) Restringendo f a Γ_1 abbiamo che $z=4$ e

$$h(x,y)=f(x,y,4)=4x+16.$$

Studiando $4x+16$ nel cerchio $x^2+y^2\le 4$ si ha che

$$8=h(-2,0)\le h(x,y)=4x+16\le h(2,0)=24.$$

Dunque i punti rilevanti in Γ_1 sono

$(-2,0,4)$ e $(2,0,4)$ $\boxed{f(-2,0,4)=8\ ,\ f(2,0,4)=24}$

2) Su Γ_2 usiamo i moltiplicatori con

$$g(x,y,z)=x^2+y^2-z=0.$$

Allora $\nabla g(x,y,z)=(2x,2y,-1)\neq(0,0,0)$ e tutti i punti di Γ_2 sono regolari

$$\begin{cases}4=\lambda 2x \to \lambda\neq 0\\ 0=\lambda 2y \to y=0\\ 2z=\lambda(-1)\\ x^2+y^2=z\end{cases}\qquad \begin{cases}y=0\\ x=2/\lambda\\ z=-\lambda/2\\ \frac{4}{\lambda^2}+0^2=-\frac{\lambda}{2} \to \lambda^3=-8\to\lambda=-2\end{cases}$$

ok perché $z<4$

$x=-1,\ z=1$

L'unico punto stazionario è $(-1,0,1)\in\Gamma_2$ e $\boxed{f(-1,0,1)=-3}$. Quindi

$(-1,0,1)$ punto di min. assoluto

$(2,0,4)$ punto di max. assoluto

2.a

$$\iint_D \frac{e^{x/y}}{y^4}\,dx\,dy \qquad D=[0,2]\times[1,2]$$

$$= \int_{y=1}^{2} \frac{1}{y^4}\left(\int_{x=0}^{2} e^{x/y}\,dx\right)dy = \int_{y=1}^{2}\frac{1}{y^4}\left[y\,e^{x/y}\right]_0^2 dy = \int_1^2 \frac{e^{2/y}-1}{y^3}\,dy$$

$t=\frac{1}{y}$

$dt=-\frac{dy}{y^2}$

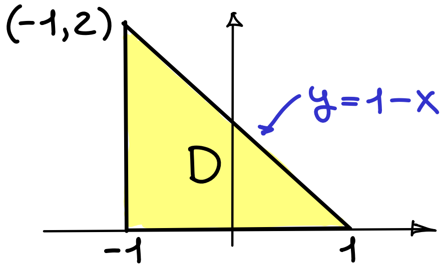
$$= \int_1^{1/2} \frac{e^{2t}-1}{1/t}\,(-dt) = \int_{1/2}^{1} \left(t e^{2t}-t\right)dt$$

$$= \left[\frac{t e^{2t}}{2}-\frac{e^{2t}}{4}-\frac{t^2}{2}\right]_{1/2}^{1} = \frac{e^2}{2}-\frac{e^2}{4}-\frac{1}{2}-\left(\cancel{\frac{e}{4}}-\cancel{\frac{e}{4}}-\frac{1}{8}\right)=\frac{e^2}{4}-\frac{3}{8}$$

dove $\int t e^{2t}\,dt=\int t\,d\left(\frac{e^{2t}}{2}\right)=\frac{t e^{2t}}{2}-\frac{e^{2t}}{4}+c.$

2.b

$$\iint_D (1-|x|-y)\,dx\,dy \qquad D=\left\{(x,y):\ \begin{matrix} x\geq -1,\ y\geq 0 \\ x+y\leq 1\end{matrix}\right\}$$



$$= \int_{x=-1}^{1}\left(\int_{y=0}^{1-x}(1-|x|-y)\,dy\right)dx$$

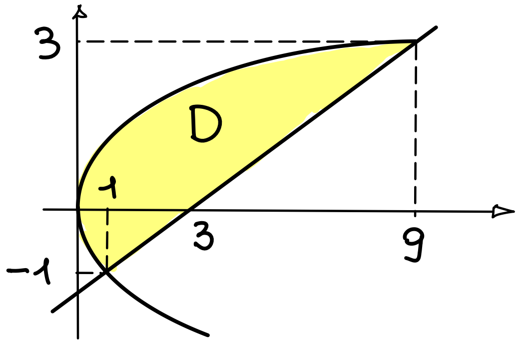
$$= \int_{-1}^{1}\left[(1-|x|)y-\frac{y^2}{2}\right]_0^{1-x}dx = \int_{-1}^{1}\left((1-|x|)(1-x)-\frac{1}{2}(1-x)^2\right)dx$$

$$\underbrace{1}_{p}-\underbrace{|x|}_{p}-\cancel{x}+\underbrace{x|x|}_{d}-\underbrace{\frac{1}{2}}_{p}+\cancel{x}-\underbrace{\frac{x^2}{2}}_{p}$$

$$= 2\int_0^1\left(\frac{1}{2}-x-\frac{x^2}{2}\right)dx=\left[x-x^2-\frac{x^3}{3}\right]_0^1=-\frac{1}{3}.$$

2.c

$$\iint_D \frac{1}{(2y-x+8)^2}\,dxdy \qquad D=\{(x,y): y^2\le x\le 2y+3\}$$



$$=\int_{y=-1}^{3}\left(\int_{x=y^2}^{2y+3}\frac{1}{(2y-x+8)^2}\,dx\right)dy$$

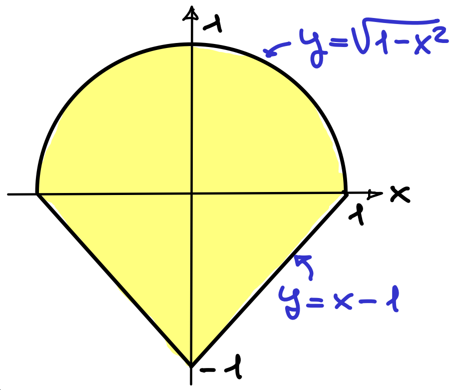
$$=\int_{y=-1}^{3}\left[\frac{1}{2y-x+8}\right]_{y^2}^{2y+3}dy=\int_{-1}^{3}\Big(\frac{1}{5}-\frac{1}{\underbrace{2y-y^2+8}_{-(y+2)(y-4)}}\Big)dy$$

$$=\frac{3-(-1)}{5}+\frac{1}{6}\int_{-1}^{3}\Big(\frac{1}{y-4}-\frac{1}{y+2}\Big)dy=\frac{4}{5}+\frac{1}{6}\left[\log\left|\frac{y-4}{y+2}\right|\right]_{-1}^{3}$$

$$=\frac{4}{5}+\frac{1}{6}\Big(\log\Big(\frac{1}{5}\Big)-\log(5)\Big)=\frac{4}{5}-\frac{\log(5)}{3}$$

2.d

$$\iint_D \frac{x+y}{1+x^2}\,dxdy \qquad D=\{(x,y): y\ge |x|-1,\ x^2+y^2\le 1\}$$



D è simmetrico rispetto a $x=0$.

$$=\iint_D\Big(\underbrace{\frac{x}{1+x^2}}_{x\text{-disp}}+\underbrace{\frac{y}{1+x^2}}_{x\text{-pari}}\Big)dxdy$$

$$=0+2\iint_{D\cap\{x\ge 0\}}\frac{y}{1+x^2}\,dxdy=2\int_{x=0}^{1}\frac{1}{1+x^2}\left(\int_{x-1}^{\sqrt{1-x^2}}y\,dy\right)dx$$

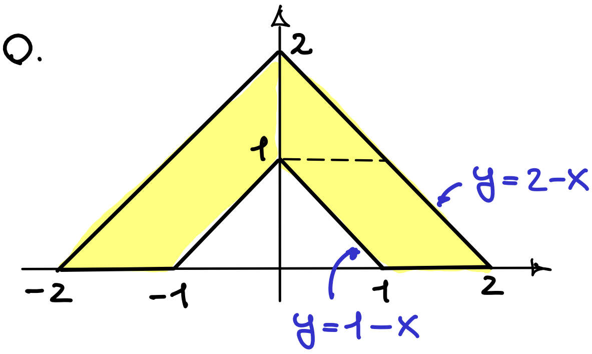
$$= \cancel{2}\int_0^1 \frac{1}{1+x^2}\left[\frac{y^2}{\cancel{2}}\right]_{x-1}^{\sqrt{1-x^2}} dx = \int_0^1 \frac{\cancel{1}-x^2-x^2+2x-\cancel{1}}{1+x^2}\,dx$$

$$= \int_0^1 \left(\frac{2x}{1+x^2} - 2\,\frac{x^2\pm 1}{1+x^2}\right) dx = \Big[\log(1+x^2) - 2x + 2\,\mathrm{arctg}(x)\Big]_0^1$$

$$= \log 2 - 2 + \frac{\pi}{2}.$$

$\boxed{2.e}$ $\displaystyle\iint_D y(1-\mathrm{sen}(x))\,dx\,dy \qquad D=\{(x,y): y\geq 0,\ 1\leq |x|+y\leq 2\}$

D è simmetrico rispetto a $x=0$.



$$= \underbrace{\iint_D y\,dx\,dy}_{\text{pari}} - \underbrace{\iint_D y\,\mathrm{sen}(x)\,dx\,dy}_{x\text{-dispari}}$$

$$= 2\iint_{D\cap\{x\geq 0\}} y\,dx\,dy + 0$$

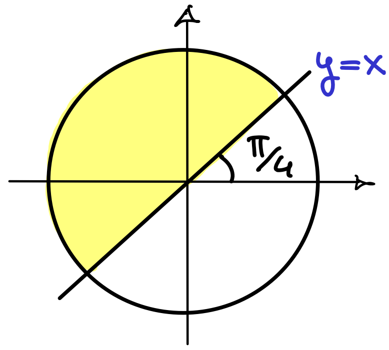
$$= 2\int_{y=0}^{1} y\left(\int_{x=1-y}^{2-y} dx\right) dy + 2\int_{y=1}^{2} y\left(\int_{x=0}^{2-y} dx\right) dy$$

$$= 2\int_0^1 y(2-\cancel{y}-1+\cancel{y})\,dy + 2\int_1^2 y(2-y)\,dy$$

$$= \left[y^2\right]_0^1 + 2\left[y^2 - \frac{y^3}{3}\right]_1^2 = 1 + \frac{4}{3} = \frac{7}{3}.$$

2.f

$$\iint_D \sqrt{1-x^2-y^2}\,dx\,dy \qquad D=\{(x,y): y\geq x,\ x^2+y^2\leq 1\}$$



$$\overset{CP}{=} \int_{\rho=0}^{1} \sqrt{1-\rho^2}\left(\int_{\theta=\frac{\pi}{4}}^{\pi+\frac{\pi}{4}} d\theta\right)\rho\,d\rho$$

$t=1-\rho^2 \quad dt=-2\rho\,d\rho$

$$=\pi\int_0^1 \sqrt{1-\rho^2}\,\rho\,d\rho=\pi\int_0^1 \sqrt{t}\cdot\frac{dt}{2}$$

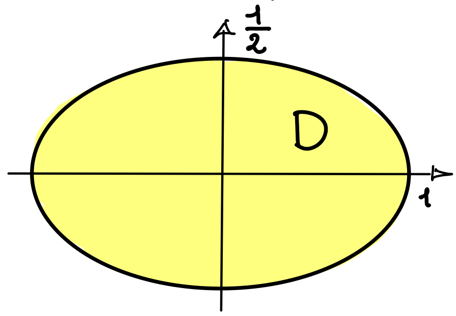
$$=\frac{\pi}{2}\left[\frac{2}{3}t^{3/2}\right]_0^1=\frac{\pi}{3}.$$

2.g

$$\iint_D (x-1)^2\,dx\,dy \qquad D=\{(x,y): x^2+4y^2\leq 1\}$$

$\underset{P}{x^2}\underset{d}{-2x}\underset{P}{+1}$

simmetrico rispetto a $x=0$



$$=\iint_D (x^2+1)\,dx\,dy$$

Con il cambio di coordinate

$$\vec{\Phi}: \begin{cases} x=\rho\cos\theta \\ y=\frac{1}{2}\rho\,\text{sen}\,\theta \end{cases} \qquad \text{si ha che } \vec{\Phi}^{-1}(D)=\overset{\rho}{[0,1]}\times\overset{\theta}{[0,2\pi]}$$

$$\text{e} \quad \left|\det J_{\vec{\Phi}}(\rho,\theta)\right|=\left|\det\begin{bmatrix}\cos\theta & -\rho\,\text{sen}\,\theta \\ \frac{1}{2}\text{sen}\,\theta & \frac{1}{2}\rho\cos\theta\end{bmatrix}\right|=\frac{1}{2}\rho.$$

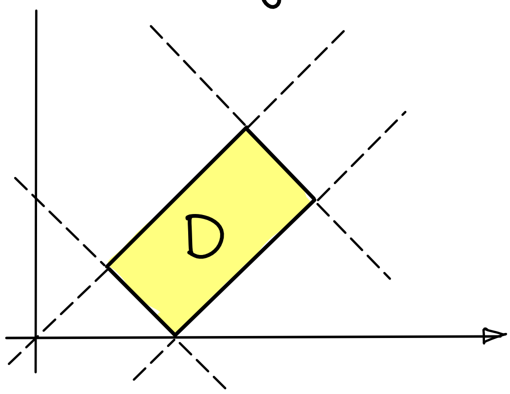
Così l'integrale diventa

$$=\int_0^{2\pi}\int_0^1 (\rho^2\cos^2\theta+1)\frac{1}{2}\rho\,d\rho\,d\theta=\frac{1}{2}\int_0^{2\pi}\left[\frac{\rho^4}{4}\cos^2\theta+\frac{\rho^2}{2}\right]_0^1 d\theta$$

$$=\frac{1}{8}\int_0^{2\pi}\cos^2\theta\,d\theta+\frac{2\pi}{4}=\frac{5\pi}{8}. \qquad \int_0^{2\pi}\cos^2\theta\,d\theta=\pi$$

$\boxed{2.h}$

$$\iint_D (x-y)\log(x+y)\,dx\,dy \qquad D=\left\{(x,y):\ \begin{matrix}0\le x-y\le 1\\ 1\le x+y\le 3\end{matrix}\right\}$$



Con il cambio di coordinate

$$\vec{\Phi}^{-1}:\begin{cases}u=x-y\\ v=x+y\end{cases} \Longrightarrow \vec{\Phi}:\begin{cases}x=\dfrac{u+v}{2}\\ y=\dfrac{-u+v}{2}\end{cases}$$

si ha che $\vec{\Phi}^{-1}(D)=\overset{u}{[0,1]}\times\overset{v}{[1,3]}$ e

$$\left|\det J_{\vec{\Phi}}(u,v)\right| = \left|\det\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}\right| = \frac{1}{2}.$$

Così l'integrale diventa

$$=\int_{u=0}^{1}\int_{v=1}^{3} u\log(v)\frac{1}{2}\,du\,dv = \frac{1}{2}\left[\frac{u^2}{2}\right]_0^1\left[v\log(v)-v\right]_1^3$$

$$=\frac{1}{4}\left(3\log 3-3-(0-1)\right)=\frac{1}{4}\left(3\log 3-2\right).$$

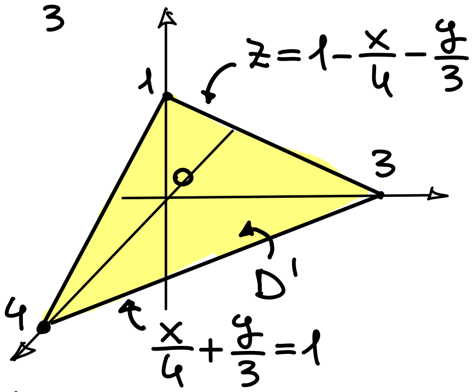
$\boxed{3.a}$ $D=\left\{(x,y,z): \frac{|x|}{4}+\frac{|y|}{3}+|z|\leq 1\right\}$ Volume?

D è simmetrico rispetto ai piani $x=0, y=0, z=0$ pertanto $|D|=8|E|$ dove

$$E=\left\{(x,y,z): \frac{x}{4}+\frac{y}{3}+z\leq 1,\ x,y,z\geq 0\right\}$$

1) Per fili con $\varphi_1(x,y)=0$, $\varphi_2(x,y)=1-\frac{x}{4}-\frac{y}{3}$

e $D'=\left\{(x,y): \frac{x}{4}+\frac{y}{3}\leq 1, x,y\geq 0\right\}$.



$$|D|=8|E|=8\iint_{D'}\left(1-\frac{x}{4}-\frac{y}{3}\right)dxdy$$

$$=8\int_{x=0}^{4}\left(\int_{y=0}^{3-\frac{3x}{4}}\left(1-\frac{x}{4}-\frac{y}{3}\right)dy\right)dx \underset{\begin{cases}x=4u\\ y=3v\end{cases}}{=} 8\int_{u=0}^{1}\left(\int_{v=0}^{1-u}(1-u-v)3\,dv\right)4\,du$$

$$=8\cdot 12\int_0^1\left[v-uv-\frac{v^2}{2}\right]_0^{1-u}du=8\cdot 12\int_0^1\left((1-u)^2-\frac{(1-u)^2}{2}\right)du$$

$$=\frac{8\cdot 12}{2}\int_0^1(1-u)^2du=48\left[-\frac{(1-u)^3}{3}\right]_0^1=\frac{48}{3}=16.$$

2) Per sezioni con $z\in[0,1]$ e $S_z=\left\{(x,y): \frac{x}{4}+\frac{y}{3}\leq 1-z\right\}$

triangolo rettangolo con cateti $4(1-z)$ e $3(1-z)$

$$|D|=8|E|=8\int_{z=0}^{1}\left(\iint_{S_z}1\,dxdy\right)dz$$

$$=8\int_0^1|S_z|dz=8\int_0^1\frac{4\cdot 3(1-z)^2}{2}dz=8\cdot 6\left[-\frac{(1-z)^3}{3}\right]_0^1$$

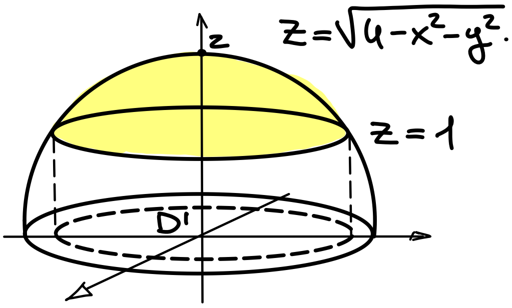
$$=8\cdot\cancel{6}^{\,2}\,\frac{1}{\cancel{3}}=16.$$

$\boxed{3.b}$ $D=\{(x,y,z): x^2+y^2+z^2\leq 4,\ z\geq 1\}$ Volume?

calotta

1) Per fili con $D'=\{x^2+y^2+1\leq 4\}$

$\varphi_1(x,y)=1$ e $\varphi_2(x,y)=\sqrt{4-x^2-y^2}$.



$$|D|=\iint_{D'}\left(\sqrt{4-x^2-y^2}-1\right)dx\,dy$$

$$\overset{CP}{=}\int_{\rho=0}^{\sqrt3}\int_{\theta=0}^{2\pi}\left(\sqrt{4-\rho^2}-1\right)\rho\,d\rho\,d\theta=2\pi\int_0^{\sqrt3}\left(\sqrt{4-\rho^2}-1\right)\rho\,d\rho$$

$t=4-\rho^2$
$dt=-2\rho\,d\rho$

$$=\cancel{2}\pi\int_4^1\left(t^{1/2}-1\right)\left(-\frac{dt}{\cancel{2}}\right)=\pi\left[\frac{2t^{3/2}}{3}-t\right]_1^4$$

$$=\pi\left(\left(\frac{16}{3}-4\right)-\left(\frac{2}{3}-1\right)\right)=\frac{5}{3}\pi.$$

2) Per sezioni con $z\in[1,2]$ e $S_z=\{(x,y): x^2+y^2\leq 4-z^2\}$

cerchio di raggio $\sqrt{4-z^2}$

$$|D|=\int_{z=1}^{2}\left(\iint_{S_z}1\,dx\,dy\right)dz=\int_1^2|S_z|\,dz$$

$$=\int_1^2\pi(4-z^2)\,dz=\pi\left[4z-\frac{z^3}{3}\right]_1^2$$

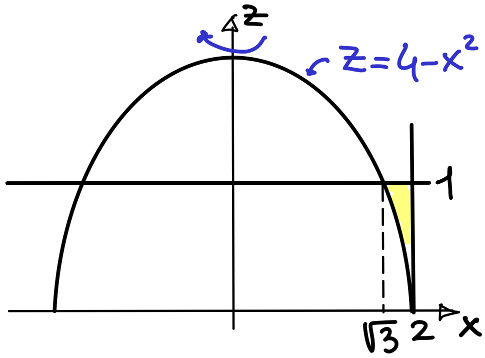
$$=\pi\left(8-\frac{8}{3}-4+\frac{1}{3}\right)=\frac{5}{3}\pi.$$

3.c $D=\{(x,y,z): 0\le 4-x^2-y^2\le z\le 1\}$ Volume?

D è ottenuto ruotando intorno all'asse z la regione piana

$\{(x,z), \sqrt{3}\le x\le 2, 4-x^2\le z\le 1\}$

$\begin{cases} z=4-x^2 \\ z=1, x\ge 0 \end{cases} \Rightarrow x=\sqrt{3}$



$$|D| \overset{CC}{=} \int_{\theta=0}^{2\pi} d\theta \cdot \int_{\rho=\sqrt{3}}^{2} \rho \left(\int_{z=4-\rho^2}^{1} dz\right) d\rho = 2\pi \int_{\sqrt{3}}^{2} \rho \left[z\right]_{4-\rho^2}^{1} d\rho$$

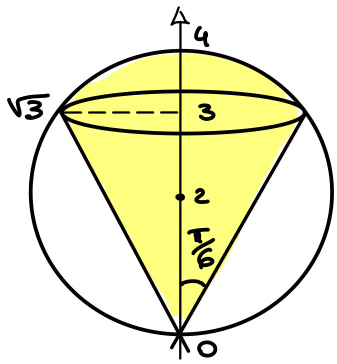
$$= 2\pi \int_{\sqrt{3}}^{2} \rho(1-(4-\rho^2))\,d\rho \overset{\substack{t=4-\rho^2 \\ dt=-2\rho d\rho}}{=} \cancel{2}\pi \int_0^1 (1-t)\frac{dt}{\cancel{2}} = \pi\left[t-\frac{t^2}{2}\right]_0^1 = \frac{\pi}{2}.$$

3.d $D=\left\{(x,y,z): \begin{array}{l} 3(x^2+y^2)\le z^2 \\ x^2+y^2+(z-2)^2\le 4 \end{array}\right\}$ Volume?

Coordinate sferiche:

$$\vec{\Phi}^{-1}(D)=\left\{(\rho,\theta,\varphi), \begin{array}{l} \varphi\in[0,\frac{\pi}{6}] \\ \theta\in[0,2\pi) \\ \rho\in[0,4\cos\varphi] \end{array}\right\}$$

$x^2+y^2+z^2-4z+\cancel{4}\le\cancel{4}$,
$\rho^2-4\rho\cos\varphi\le 0, \rho\le 4\cos\varphi$



$$|D| = \iiint_{\vec{\Phi}^{-1}(D)} \rho^2 \operatorname{sen}\varphi\, d\rho\, d\theta\, d\varphi = 2\pi \int_{\varphi=0}^{\frac{\pi}{6}} \operatorname{sen}\varphi \left(\int_{\rho=0}^{4\cos\varphi} \rho^2 d\rho\right) d\varphi$$

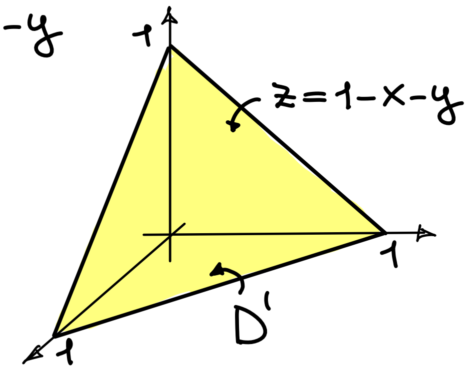
$$= 2\pi \int_0^{\frac{\pi}{6}} \operatorname{sen}\varphi \left[\frac{\rho^3}{3}\right]_0^{4\cos\varphi} d\varphi = \frac{2\cdot 4^3\pi}{3} \int_0^{\frac{\pi}{6}} (-\cos^3\varphi)\, d(\cos\varphi)$$

$$= \frac{2\cdot 4^{\cancel{3}\,2}\pi}{3}\left[-\frac{\cos^4\varphi}{\cancel{4}}\right]_0^{\frac{\pi}{6}} = \frac{32\pi}{3}\left(1-\left(\frac{\sqrt{3}}{2}\right)^4\right) = \frac{\cancel{32}^{\,2}\pi}{3}\cdot\frac{7}{\cancel{16}} = \frac{14\pi}{3}.$$

$\boxed{4.a}$ $$\iiint_D \frac{1}{(x+y+z+1)^3}\,dx\,dy\,dz$$

dove $D=\{(x,y,z): x,y,z \geq 0,\ x+y+z\leq 1\}$.

Per fili con $\varphi_1(x,y)=0$, $\varphi_2(x,y)=1-x-y$
e $D'=\{(x,y): x,y\geq 0,\ x+y\leq 1\}$



$$=\iint_{D'}\left(\int_{z=0}^{1-x-y}\frac{1}{(x+y+z+1)^3}\,dz\right)dx\,dy$$

$$=\iint_{D'}\left[\frac{-1/2}{(x+y+z+1)^2}\right]_0^{1-x-y}dx\,dy=\frac{1}{2}\iint_{D'}\left(\frac{1}{(x+y+1)^2}-\frac{1}{4}\right)dx\,dy$$

$$=\frac{1}{2}\int_{x=0}^{1}\left(\int_{y=0}^{1-x}\frac{1}{(x+y+1)^2}\,dy\right)dx-\frac{|D'|^{=\frac{1}{2}}}{8}$$

$$=\frac{1}{2}\int_0^1\left[-\frac{1}{x-y+1}\right]_0^{1-x}dx-\frac{1}{16}=\frac{1}{2}\int_0^1\left(\frac{1}{x+1}-\frac{1}{2}\right)dx-\frac{1}{16}$$

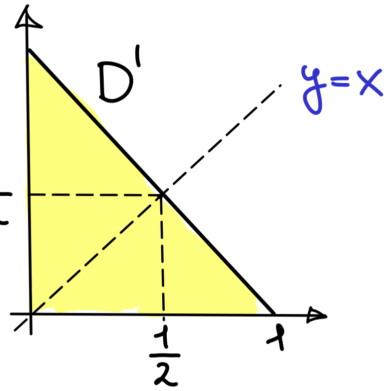
$$=\frac{1}{2}\Big[\log(x+1)\Big]_0^1-\frac{1}{4}-\frac{1}{16}=\frac{\log(2)}{2}-\frac{5}{16}.$$

$\boxed{4.b}$ $\displaystyle\iiint_D |y-x| e^z \, dx\,dy\,dz$

dove $D=\{(x,y,z): x,y,z\in[0,1],\ z\le x+y\le 1\}$.

Per fili con $D'=\{(x,y): x,y\ge 0 \text{ e } x+y\le 1\}$ e

$\varphi_1(x,y)=0$, $\varphi_2(x,y)=x+y$.



$$= \iint_{D'} |y-x| \left(\int_{z=0}^{x+y} e^z \, dz \right) dx\,dy$$

$$= \iint_{D'} |y-x| (e^{x+y}-1)\, dx\,dy$$

$$= \int_{x=0}^{1/2} \left(\int_{y=x}^{1-x} (y-x)(e^{x+y}-1)\, dy \right) dx + \int_{y=0}^{1/2} \left(\int_{x=y}^{1-y} (x-y)(e^{x+y}-1)\, dx \right) dy$$

UGUALI

$$= 2\int_{x=0}^{1/2} \left[(y-x-1)e^{x+y} - \frac{y^2}{2} + xy \right]_x^{1-x} dx$$

$$= 2\int_{x=0}^{1/2} \left(-2xe - \frac{(1-x)^2}{2} + x(1-x) - \left(-e^{2x} \overbrace{- \frac{x^2}{2} + x^2}^{\frac{x^2}{2}} \right) \right) dx$$

$$= \left[-2ex^2 + \frac{(1-x)^3}{3} + x^2 - 2\frac{x^3}{3} + e^{2x} - \frac{x^3}{3} \right]_0^{1/2}$$

$$= -\frac{e}{2} + \frac{1}{24} + \frac{1}{4} - \frac{1}{8} + e - \frac{1}{3} - 1 = \frac{e}{2} - \frac{7}{6}.$$

$\boxed{4.C}$ $\displaystyle\iiint_D z e^{x^2+y^2}\,dx\,dy\,dz$

dove $D=\{(x,y,z): x,y,z\geq 0,\ x^2+y^2+z^2\leq 1\}$.

Per sezioni: per $z\in[0,1]$

$$S_z=\{(x,y): x^2+y^2\leq 1-z^2,\ x\geq 0, y\geq 0\}.$$

così l'integrale diventa

$$=\int_{z=0}^{1} z\left(\iint_{S_z} e^{x^2+y^2}\,dx\,dy\right)dz$$

$$\stackrel{CP}{=}\int_0^1 z\left(\int_{\rho=0}^{\sqrt{1-z^2}}\int_{\theta=0}^{\frac{\pi}{2}} e^{\rho^2}\rho\,d\rho\,d\theta\right)dz$$

$$=\frac{\pi}{2}\int_0^1 z\left[\frac{e^{\rho^2}}{2}\right]_0^{\sqrt{1-z^2}}dz=\frac{\pi}{4}\int_0^1\left(e^{1-z^2}-1\right)z\,dz$$

$$=\frac{\pi e}{4}\int_0^1 e^{-z^2}z\,dz-\frac{\pi}{4}\cdot\left[\frac{z^2}{2}\right]_0^1$$

$$=\frac{\pi e}{4}\left[-\frac{e^{-z^2}}{2}\right]_0^1-\frac{\pi}{8}$$

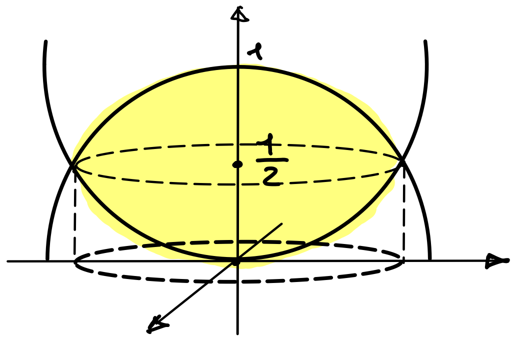
$$=\frac{\pi e}{8}\left(-e^{-1}+1\right)-\frac{\pi}{8}=\frac{\pi}{8}(e-2).$$

4.d $\displaystyle\iiint_D z^2\,dx\,dy\,dz$

dove $D=\{(x,y,z): x^2+y^2\leq \min(1-z^2, 2z-z^2)\}$

$$\begin{cases} x^2+y^2\leq 1-z^2 \\ x^2+y^2\leq 2z-z^2 \end{cases} \Longleftrightarrow \begin{cases} x^2+y^2+z^2\leq 1 \\ x^2+y^2+(z-1)^2\leq 1 \end{cases}$$

D è l'intersezione di due palle di raggio 1 e centri $(0,0,0)$ e $(0,0,1)$



In coordinate cilindriche l'integrale si spezza sui due "gusci"

$$=\int_{z=0}^{\frac12} z^2\left(\int_{\theta=0}^{2\pi}\left(\int_{\rho=0}^{\sqrt{2z-z^2}}\rho\,d\rho\right)d\theta\right)dz+\int_{z=\frac12}^{1} z^2\left(\int_{\theta=0}^{2\pi}\left(\int_{\rho=0}^{\sqrt{1-z^2}}\rho\,d\rho\right)d\theta\right)dz$$

$$=\int_0^{\frac12} z^2\cdot\cancel{2}\pi\cdot\frac{2z-z^2}{\cancel{2}}\,dz+\int_{\frac12}^{1} z^2\cdot\cancel{2}\pi\cdot\frac{1-z^2}{\cancel{2}}\,dz$$

$$=\pi\left[\frac{\cancel{2}z^4}{\cancel{4}\,2}-\frac{z^5}{5}\right]_0^{\frac12}+\pi\left[\frac{z^3}{3}-\frac{z^5}{5}\right]_{\frac12}^{1}$$

$$=\pi\left(\frac{1}{32}+\frac13-\frac15-\frac{1}{24}\right)=\frac{\pi 59}{480}.$$

$$\boxed{4.e} \qquad \iiint_D |x-1| \, dx\,dy\,dz$$

dove $D = \{(x,y,z) : x^2+y^2+z^2 \leq 4,\ x \geq 0,\ z \geq 0\}$.

$$= \int_{x=0}^{2} |x-1| \Bigg(\iint_{\substack{\{y^2+z^2 \leq 4-x^2\} \\ z \geq 0}} dy\,dz \Bigg) dx = \int_0^2 |x-1| \cdot \underbrace{\left(\frac{\pi}{2}(4-x^2)\right)}_{\text{Area}} dx$$

semicerchio di raggio $\sqrt{4-x^2}$

$$= \frac{\pi}{2}\left(\int_0^1 (1-x)(4-x^2)\,dx - \int_1^2 (1-x)(4-x^2)\,dx \right)$$

$$\to 4 - 4x - x^2 + x^3 \xrightarrow{\int} 4x - 2x^2 - \frac{x^3}{3} + \frac{x^4}{4}$$

$$= \frac{\pi}{2}\left(2\left(\cancel{4} - \cancel{2} - \frac{1}{3} + \frac{1}{4}\right) - 0 - \left(\cancel{8} - \cancel{8} - \frac{8}{3} + \cancel{4}\right) \right)$$

$$= \frac{15\pi}{12} = \frac{5\pi}{4}.$$