ANALISI MATEMATICA 2 - FOGLIO 4

1.a $2y-x^2=2+\log(y)$ in $(0,1)$. T_2 di $y=\varphi(x)$ in $x=0$?

Sia $f(x,y)=2y-x^2-2-\log(y)$.

$$f_y(0,1)=\left(2-2x-\frac{1}{y}\right)\Big|_{(0,1)}=1\neq 0 \overset{\text{TFI}}{\Longrightarrow} \exists\, y=\varphi(x).$$

$$\varphi(0)=1,\quad \varphi'(0)=-\frac{f_x(0,1)}{f_y(0,1)}=-\frac{(-2x)}{1}\Big|_{(0,1)}=0.$$

Inoltre

$$2\varphi(x)-x^2=2+\log(\varphi(x)) \overset{D}{\Longrightarrow} 2\varphi'(x)-2x=\frac{\varphi'(x)}{\varphi(x)}$$

$$\overset{D}{\Longrightarrow} 2\varphi''(x)-2=\frac{\varphi''(x)\varphi(x)-(\varphi'(x))^2}{(\varphi(x))}$$

Per $x=0$ otteniamo

$$2\varphi''(0)-2=\frac{\varphi''(0)\cdot 1-0^2}{1^2} \Longrightarrow \varphi''(0)=2.$$

Quindi

$$T_2(x)=1+0\cdot x+\frac{2}{2}x^2=1+x^2.$$

1.b $\log(1+x^2-2y)=x+y^4$ in $(0,0)$. T_2 di $y=\varphi(x)$ in $x=0$?

Sia $f(x,y)=\log(1+x^2-2y)-x-y^4$.

$$f_y(0,0)=\left(\frac{-2}{1+x^2-2y}-4y^3\right)\Big|_{(0,0)}=-2\neq 0 \overset{\text{TFI}}{\Longrightarrow} \exists\, y=\varphi(x)$$

$$\varphi(0)=0 \text{ e } \varphi'(0)=-\frac{f_x(0,0)}{f_y(0,0)}=-\frac{\left(\frac{2x}{1+x^2-2y}-1\right)\Big|_{(0,0)}}{(-2)}=-\frac{1}{2}.$$

Inoltre

$$\log(1+x^2-2\varphi(x))=x+(\varphi(x))^4 \overset{D}{\Longrightarrow} \frac{2x-2\varphi'(x)}{1+x^2-2\varphi(x)}=1+4\varphi^3(x)\varphi'(x)$$

$$\overset{D}{\Longrightarrow} \frac{1}{(1+x^2-2\varphi(x))^2}\Big[(2-2\varphi''(x))(1+x^2-2\varphi(x))-(2x-2\varphi'(x))^2\Big]$$
$$=12\varphi^2(x)(\varphi'(x))^2+4\varphi^3(x)\varphi''(x)$$

Per $x=0$ si ha
$$\frac{1}{1^2}\left[\left(2-2\varphi''(0)\right)\cdot 1-\left(-2\cdot\left(-\frac{1}{2}\right)\right)^2\right]=0$$
$$2-2\varphi''(0)-1=0 \implies \varphi''(0)=\frac{1}{2}.$$
Quindi
$$T_2(x)=0-\frac{x}{2}+\frac{1}{2}\cdot\frac{1}{2}x^2=-\frac{x}{2}+\frac{x^2}{4}.$$

$\boxed{1.c}$ $y(x^2+y^2)=2x^2$ in $(1,1)$. T_2 di $y=\varphi(x)$ in $x=1$?

Sia $f(x,y)=y(x^2+y^2)-2x^2=yx^2+y^3-2x^2$.

$$f_y(1,1)=(x^2+3y^2)|_{(1,1)}=4\neq 0 \overset{\text{TFI}}{\implies} \exists\, y=\varphi(x).$$
$$\varphi(1)=1,\ \varphi'(1)=-\frac{f_x(1,1)}{f_y(1,1)}=-\frac{(2yx-4x)|_{(1,1)}}{4}=\frac{1}{2}.$$
Inoltre $\varphi(x)(x^2+\varphi^2(x))=2x^2$
$$\overset{D}{\implies} \varphi'(x)(x^2+\varphi^2(x))+\varphi(x)(2x+2\varphi(x)\varphi'(x))=4x$$
$$\varphi'(x)x^2+\varphi(x)\cdot 2x+3\varphi^2(x)\varphi'(x)=4x$$
$$\overset{D}{\implies} \varphi''(x)x^2+\varphi'(x)\cdot 2x+\varphi'(x)2x+2\varphi(x)$$
$$+6\varphi(x)(\varphi'(x))^2+3\varphi^2(x)\varphi''(x)=4$$
Per $x=1$,
$$\varphi''(1)+\cancel{1}+\cancel{1}+\cancel{2}+6\cdot\frac{1}{4}+3\varphi''(1)=\cancel{4} \implies \varphi''(1)=-\frac{3}{8}.$$
Quindi
$$T_2(x)=1+\frac{1}{2}(x-1)-\frac{3}{16}(x-1)^2.$$

$\boxed{1.d}$ $x^2-2x+4y^2=3$ in $(1,1)$. T_2 di $y=\varphi(x)$ in $x=1$?

Sia $f(x,y)=x^2-2x+4y^2-3$.

$$f_y(1,1)=(8y)|_{(1,1)}=8\neq 0 \overset{\text{TFI}}{\implies} \exists\, y=\varphi(x).$$

$$\varphi(1)=1,\ \varphi'(1)=-\frac{f_x(1,1)}{f_y(1,1)}=-\frac{(2x-2)_{|(1,1)}}{8}=0.$$

Inoltre $x^2-2x+4\varphi^2(x)=3$

$$\overset{D}{\Longrightarrow}\ 2x-2+8\varphi(x)\cdot\varphi'(x)=0$$

$$\overset{D}{\Longrightarrow}\ 2+8\left(\varphi'(x)\right)^2+8\varphi(x)\varphi''(x)=0.$$

Per $x=1$,

$$2+8\cdot 0+8\cdot 1\cdot\varphi''(1)=0\ \Longrightarrow\ \varphi''(1)=-\frac{1}{4}.$$

Quindi

$$T_2(x)=1-\frac{1}{8}(x-1)^2.$$

$\boxed{1.e}$ $x^2y=e^{xy+1}$ in $(-1,1)$. T_2 di $y=\varphi(x)$ in $x=-1$?

Sia $f(x,y)=x^2y-e^{xy+1}$.

$$f_y(-1,1)=\left(x^2-e^{xy+1}\cdot x\right)_{|(-1,1)}=2\neq 0\ \overset{TFI}{\Longrightarrow}\ \exists\, y=\varphi(x).$$

$$\varphi(-1)=1,\ \varphi'(-1)=-\frac{f_x(-1,1)}{f_y(-1,1)}=-\frac{\left(2xy-e^{xy+1}y\right)_{|(-1,1)}}{2}=\frac{3}{2}.$$

Inoltre $x^2\varphi(x)=e^{x\varphi(x)+1}$

$$\overset{D}{\Longrightarrow}\ 2x\varphi(x)+x^2\varphi'(x)=e^{x\varphi(x)+1}\left(\varphi(x)+x\varphi'(x)\right)$$

$$\overset{D}{\Longrightarrow}\ 2\varphi(x)+4x\varphi'(x)+x^2\varphi''(x)$$

$$=e^{x\varphi(x)+1}\left[\left(\varphi(x)+x\varphi'(x)\right)^2+\left(2\varphi'(x)+x\varphi''(x)\right)\right]$$

Per $x=-1$,

$$2\cdot 1+4(-1)\cdot\frac{3}{2}+1\cdot\varphi''(-1)=1\left[\left(1-\frac{3}{2}\right)^2+\left(2\cdot\frac{3}{2}-1\cdot\varphi''(-1)\right)\right]$$

da cui

$$\varphi''(-1)=\frac{1}{2}\left(\frac{1}{4}+3-2+6\right)=\frac{29}{8}.$$

Quindi

$$T_2(x)=1+\frac{3}{2}(x+1)+\frac{29}{16}(x+1)^2.$$

$\boxed{1.f}$ $x^5+3x^2y-2y^4=1$ in $(1,0)$. T_2 di $y=\varphi(x)$ in $x=1$?

Sia $f(x,y)=x^5+3x^2y-2y^4-1$.

$$f_y(1,0)=(3x^2-8y^3)_{|(1,0)}=3\neq 0 \overset{\text{TFI}}{\Longrightarrow} \exists\, y=\varphi(x).$$

$$\varphi(1)=0,\ \varphi'(1)=-\frac{f_x(1,0)}{f_y(1,0)}=-\frac{(5x^4+6xy)}{3}_{|(1,0)}=-\frac{5}{3}.$$

Inoltre $x^5+3x^2\varphi(x)-2\varphi^4(x)=1$

$$\overset{D}{\Longrightarrow} 5x^4+6x\varphi(x)+3x^2\varphi'(x)-8\varphi^3(x)\cdot\varphi'(x)=0$$

$$\overset{D}{\Longrightarrow} 20x^3+6\varphi(x)+12x\varphi'(x)+3x^2\varphi''(x)$$
$$-24\varphi^2(x)\cdot\varphi'(x)-8\varphi^3(x)\varphi''(x)=0$$

Per $x=1$,

$$\cancel{20}+0+\cancel{12\left(-\frac{5}{3}\right)}+3\varphi''(1)=0 \Longrightarrow \varphi''(1)=0.$$

Quindi

$$T_2(x)=-\frac{5}{3}(x-1).$$

$\boxed{2.a}$ $\begin{cases} xyz=-2 \\ x^3+y^3+z^3=8 \end{cases}$ in $(-1,2,1)$. T_1 di $y=\varphi(x)$ e $z=\psi(x)$ in $x=-1$?

Siano $f(x,y,z)=xyz+2=0$, $g(x,y,z)=x^3+y^3+z^3-8=0$.

$$J=\begin{bmatrix} f_y & f_z \\ g_y & g_z \end{bmatrix}=\begin{bmatrix} xz & xy \\ 3y^2 & 3z^2 \end{bmatrix} \xRightarrow{(-1,2,1)} \begin{bmatrix} -1 & -2 \\ 12 & 3 \end{bmatrix} \Rightarrow \det(J)=21\neq 0$$

Quindi per TFI $\exists\, y=\varphi(x)$ e $\exists\, z=\psi(x)$: $\varphi(-1)=2$, $\psi(-1)=1$.

Inoltre derivando

$$\begin{cases} x\varphi(x)\psi(x)=-2 \\ x^3+(\varphi(x))^3+(\psi(x))^3=8 \end{cases}$$

si ottiene

$$\begin{cases} \varphi(x)\psi(x)+x\varphi'(x)\psi(x)+x\varphi(x)\psi'(x)=0 \\ 3x^2+3(\varphi(x))^2\cdot\varphi'(x)+3(\psi(x))^2\cdot\psi'(x)=0 \end{cases}$$

Per $x=-1$ si ha

$$\begin{cases} 2\cdot 1-\varphi'(-1)\cdot 1-2\psi'(-1)=0 \\ 3+12\varphi'(-1)+3\psi'(-1)=0 \end{cases} \qquad \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} \varphi'(-1) \\ \psi'(-1) \end{bmatrix}=\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

da cui

$$\varphi'(-1)=\frac{\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}}=\frac{4}{-7} \quad \text{e} \quad \psi'(-1)=\frac{\begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}}=\frac{-9}{-7}=\frac{9}{7}.$$

REGOLA DI CRAMER

Infine

$$\varphi)\ T_1(x)=2-\frac{4}{7}(x+1) \qquad \psi)\ T_1(x)=1+\frac{9}{7}(x+1).$$

$\boxed{2.b}$ $\begin{cases} x^2+y^2-z^2=1 \\ xy+yz=xz-3 \end{cases}$ in $(1,-1,1)$. T_1 di $y=\varphi(x)$ e $z=\psi(x)$ in $x=1$?

Siano

$$f(x,y,z)=x^2+y^2-z^2-1=0, \quad g(x,y,z)=xy+yz-xz+3=0$$

$$J=\begin{bmatrix} f_y & f_z \\ g_y & g_z \end{bmatrix}=\begin{bmatrix} 2y & -2z \\ x+z & y-x \end{bmatrix} \xRightarrow{(1,-1,1)} \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix} \Rightarrow \det(J)=16\neq 0$$

Quindi per TFI $\exists\, y=\varphi(x)$ e $\exists\, z=\psi(x)$: $\varphi(1)=-1$ e $\psi(1)=1$.

Inoltre derivando

$$\begin{cases} x^2+\varphi^2(x)-\psi^2(x)=1 \\ x\varphi(x)+\varphi(x)\psi(x)=x\psi(x)-3 \end{cases}$$

si ottiene

$$\begin{cases} 2x+2\varphi(x)\varphi'(x)-2\psi(x)\psi'(x)=0 \\ \varphi(x)+x\varphi'(x)+\varphi'(x)\psi(x)+\varphi(x)\psi'(x)=\psi(x)+x\psi'(x) \end{cases}$$

Per $x=1$ si ha

$$\begin{cases} 2-2\varphi'(1)-2\psi'(1)=0 \\ -1+\varphi'(1)+\varphi'(1)-\psi'(1)=1+\psi'(1) \end{cases} \qquad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \varphi'(1) \\ \psi'(1) \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

da cui

$$\varphi'(1)=\frac{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}=1 \qquad \psi'(1)=\frac{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}=0$$

Infine

$\varphi)$ $T_1(x)=-1+(x-1)=x-2$, $\quad\psi)$ $T_1(x)=1+0\cdot(x-1)=1$.

$\boxed{3.a}$ $\{(x,y,z) : xyz=-8\}$ Piano tangente in $(-2,1,4)$?

Sia $f(x,y,z)=xyz+8$. Allora

$$\nabla f(x,y,z)=(yz,xz,xy)$$

e il piano tangente in $(-2,1,4)$ è

$$f_x(-2,1,4)(x+2)+f_y(-2,1,4)(y-1)+f_z(-2,1,4)(z-4)=0$$

ossia $4(x+2)-8(y-1)-2(z-4)=0$

$$2x-4y-z+12=0.$$

$\boxed{3.b}$ $\{(x,y,z) : (x-1)^2+(y-2)^2+4(z-2)^2=9\}$

Piano tangente in $(3,1,3)$?

Sia $f(x,y,z)=(x-1)^2+(y-2)^2+4(z-2)^2-9$. Allora

$$\nabla f(x,y,z)=(2(x-1),2(y-2),8(z-2))$$

e dunque $\nabla f(3,1,3)=(4,-2,8)$.

Così il piano tangente in $(3,1,3)$ è

$$\overset{2}{\cancel{4}}(x-3)-\overset{1}{\cancel{2}}(y-1)+\overset{4}{\cancel{8}}(z-3)=0$$

ossia

$$2x-y+4z=17.$$

$\boxed{3.c}$ $\{(x,y,z) : (x^2+y^2-z^2)(x^2+y^2-8z^2)=0\}$

Piano tangente in $(2,2,1)$?

Sia $f(x,y,z)=(x^2+y^2-z^2)(x^2+y^2-8z^2)$. Allora

$$\nabla f(x,y,z)=(2x,2y,-2z)(x^2+y^2-8z^2)+(x^2+y^2-z^2)(2x,2y,-16z)$$

e quindi $\nabla f(2,2,1)=(4,4,-2)\cdot 0+7\cdot(4,4,-16)=28(1,1,-4)$.

Così il piano richiesto è

$$1\cdot(x-2)+1\cdot(y-2)-4(z-1)=0 \implies x+y-4z=0.$$

4.a $f(x,y)=xy+x^2$ Punti stazionari vincolati in $\Gamma=\{(x,y): x^2+y^2+xy=4\}$?

Sia $g(x,y)=x^2+y^2+xy-4=0$ allora

$$\nabla g(x,y)=(2x+y,\,2y+x)=(0,0) \iff (x,y)=(0,0)\notin\Gamma$$

Tutti i punti di Γ sono regolari

Punti stazionari vincolati

$$\begin{cases}\nabla f(x,y)=\lambda\nabla g(x,y)\\ g(x,y)=0\end{cases} \qquad \begin{cases} y+2x=\lambda(2x+y) \rightarrow (\lambda-1)(2x+y)=0 \\ x=\lambda(2y+x) \\ x^2+y^2+xy=4\end{cases}$$

$\lambda=1$ $\qquad y=-2x$

$$\begin{cases}\lambda=1\\ \cancel{x}=2y+\cancel{x} \rightarrow y=0\\ x^2+y^2+xy=4 \rightarrow x^2=4 \rightarrow x=\pm 2\end{cases} \Longrightarrow \boxed{\begin{matrix}(2,0)\\(-2,0)\end{matrix}} \quad \lambda=1$$

$$\begin{cases} y=-2x \rightarrow y=\mp\frac{4}{\sqrt3}\\ x=\lambda(-4x+x) \rightarrow \lambda=-\frac{1}{3}\\ x^2+4x^2-2x^2=4 \rightarrow x=\pm\frac{2}{\sqrt3}\end{cases} \Longrightarrow \boxed{\begin{matrix}\left(\frac{2}{\sqrt3},-\frac{4}{\sqrt3}\right)\\ \left(-\frac{2}{\sqrt3},\frac{4}{\sqrt3}\right)\end{matrix}} \quad \lambda=-\frac{1}{3}$$

$3x^2=4$

4.b $f(x,y)=(3x+2y)^2$ Punti stazionari vincolati in $\Gamma=\{(x,y): 4x^2+y^2=4\}$?

Sia $g(x,y)=4x^2+y^2-4=0$ allora

$$\nabla g(x,y)=(8x,\,2y)=(0,0) \iff (x,y)=(0,0)\notin\Gamma$$

Tutti i punti di Γ sono regolari

Punti stazionari vincolati:

$$\begin{cases}\nabla f(x,y)=\lambda\nabla g(x,y)\\ g(x,y)=0\end{cases} \qquad \begin{cases}\cancel{2}(3x+2y)\cdot 3=\lambda(\overset{4}{\cancel{8}}x)\\ \cancel{2}(3x+2y)\cdot 2=\lambda(\cancel{2}y)\\ 4x^2+y^2=4\end{cases}$$

$2(1)-3(2)$

$$\begin{cases} 8\lambda x - 3\lambda y = 0 \\ (3x+2y)\cdot 2 = \lambda y \\ 4x^2+y^2=4 \end{cases} \to \lambda(8x-3y)=0 \quad \lambda=0 \quad y=\frac{8}{3}x$$

$$\begin{cases} \lambda=0 \\ 3x+2y=0 \to y=-\frac{3x}{2} \to y=\mp\frac{6}{5} \\ 4x^2+y^2=4 \to 4x^2+\frac{9x^2}{4}=4 \to x=\pm\frac{4}{5} \end{cases} \qquad \boxed{\left(\frac{4}{5},-\frac{6}{5}\right) \quad \left(-\frac{4}{5},\frac{6}{5}\right)} \quad \lambda=0$$

$$\begin{cases} y=\frac{8}{3}x & \to y=\pm\frac{8}{5} \\ \left(3x+2\left(\frac{8}{3}x\right)\right)\cdot 2 = \lambda\left(\frac{8}{3}x\right) & \to \lambda=\frac{25}{4} \\ 4x^2+\frac{64}{9}x^2=4 & \to x=\pm\frac{3}{5} \end{cases} \qquad \boxed{\left(\frac{3}{5},\frac{8}{5}\right) \quad \left(-\frac{3}{5},-\frac{8}{5}\right)} \quad \lambda=\frac{25}{4}$$

4.c $f(x,y)=x-x^2-y^2$ Punti stazionari vincolati in $\Gamma=\{(x,y) : x^2+2y^2=9\}$?

Sia $g(x,y)=x^2+2y^2-9=0$ allora

$$\nabla g(x,y)=(2x,4y)=(0,0) \iff (x,y)=(0,0) \notin \Gamma$$

Tutti i punti di Γ sono regolari.

Punti stazionari vincolati:

$$\begin{cases} \nabla f(x,y)=\lambda\nabla g(x,y) \\ g(x,y)=0 \end{cases} \qquad \begin{cases} 1-2x=\lambda 2x \\ -\cancel{2}y=\lambda \cancel{4}^{2}y \to y(2\lambda+1)=0 \\ x^2+2y^2=9 \end{cases} \qquad y=0 \quad \lambda=-\frac{1}{2}$$

$$\begin{cases} y=0 \\ 1-2x=\lambda 2x \\ x^2=9 \end{cases} \cup \begin{cases} \lambda=-\frac{1}{2} \\ 1-2x=-x \to x=1 \\ x^2+2y^2=9 \to y=\pm 2 \end{cases}$$

$x=3$ $\quad x=-3$

$\lambda=-\frac{5}{6}$ $\quad \lambda=-\frac{7}{6}$

$\boxed{(3,0),\ (-3,0)}$ $\qquad \boxed{(1,2),\ (1,-2)}$

$\boxed{4.d}$ $f(x,y)=\log(xy)$ Punti stazionari vincolati in $\Gamma=\{(x,y): x^2+2y^2=y\}$?

$D=\{(x,y): xy>0\}$.

Sia $g(x,y)=x^2+2y^2-y=0$ allora

$$\nabla g(x,y)=(2x,4y-1)=(0,0) \iff (x,y)=(0,\tfrac{1}{4})\notin\Gamma$$

Tutti i punti di Γ sono regolari.

Punti stazionari vincolati:

$$\begin{cases}\nabla f(x,y)=\lambda\nabla g(x,y)\\ g(x,y)=0\end{cases} \qquad \begin{cases}\frac{1}{x}=\lambda 2x & \to\ x^2=\frac{1}{2\lambda}\\ \frac{1}{y}=\lambda(4y-1) & \to\ 4y^2-y=2x^2\\ x^2+2y^2=y & \to\ 2y^2-\frac{y}{2}+2y^2=y\end{cases}$$

$xy>0$

$4y^2=\frac{3}{2}y \quad y\neq 0$

$\to y=\frac{3}{8}$

$$\begin{cases}y=\frac{3}{8}\\ 2x^2=4\cdot\frac{9}{64}-\frac{3}{8}=\frac{3}{16} \to x=\cancel{\pm}\,\frac{\sqrt{6}}{8}\\ \lambda=\frac{16}{3}\end{cases}$$

$$\boxed{\left(\frac{\sqrt{6}}{8},\frac{3}{8}\right)}$$

$\boxed{4.e}$ $f(x,y)=x^2+(y-5)^2$ Punti stazionari vincolati in $\Gamma=\{(x,y): y^2+1=x\}$?

Sia $g(x,y)=y^2+1-x=0$ allora

$$\nabla g(x,y)=(-1,2y)=(0,0) \text{ impossibile}$$

Tutti i punti di Γ sono regolari.

Punti stazionari vincolati:

$$\begin{cases}\nabla f(x,y)=\lambda\nabla g(x,y)\\ g(x,y)=0\end{cases} \qquad \begin{cases}2x=-\lambda\\ \cancel{2}(y-5)=\cancel{2}\lambda y & \to\ y-5=-2xy \ \to y\neq 0\\ y^2+1=x & \xrightarrow{\cdot 2y}\ 2y(y^2+1)=2xy=-y+5\end{cases}$$

$2y^3+3y-5=0$

$(y-1)\underbrace{(2y^2+2y+5)}_{\Delta<0}=0$

da cui $y=1$

e $x=1^2+1=2$

$$\boxed{(2,1)}$$

$\boxed{4.f}$ $f(x,y)=2x+y$ Punti stazionari vincolati in $\Gamma=\{(x,y):(x+1)^2+y^2=5\}$?

Sia $g(x,y)=(x+1)^2+y^2-5=0$ allora

$$\nabla g(x,y)=(2(x+1),2y)=(0,0) \iff (x,y)=(-1,0)\notin\Gamma$$

Tutti i punti di Γ sono regolari.

Punti stazionari vincolati:

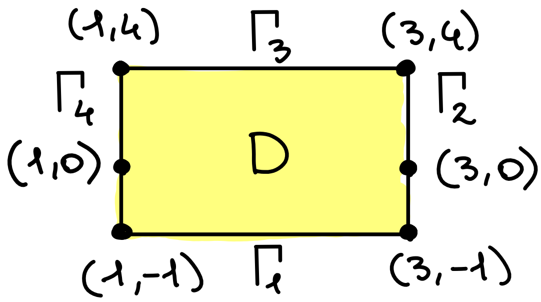
$$\begin{cases}\nabla f(x,y)=\lambda\nabla g(x,y)\\ g(x,y)=0\end{cases} \quad \begin{cases}\cancel{2}=\lambda\cancel{2}(x+1) \to x+1=\frac{1}{\lambda}\\ 1=\lambda 2y \to y=\frac{1}{2\lambda}\\ (x+1)^2+y^2=5 \to \frac{1}{\lambda^2}+\frac{1}{4\lambda^2}=5 \to \lambda=\pm\frac{1}{2}\end{cases}$$

$$\begin{cases}\lambda=\frac{1}{2}\\ x=2-1=1\\ y=1\end{cases} \qquad \begin{cases}\lambda=-\frac{1}{2}\\ x=-2-1=-3\\ y=-1\end{cases}$$

$$\boxed{(1,1),\ (-3,-1)}$$

5.a $f(x,y)=\log((x+1)^2+y^2)$ Max/min in $D=[1,3]\times[-1,4]$?

$f\in C^2(\mathbb{R}^2\setminus\{(-1,0)\})$, D è compatto e $(-1,0)\notin D$.



$\partial D=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$

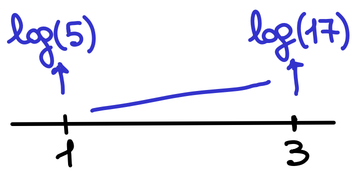
Punti stazionari interni a D:

$$\nabla f(x,y)=\left(\frac{2(x+1)}{(x+1)^2+y^2},\frac{2y}{(x+1)^2+y^2}\right)=(0,0)$$ Senza soluzioni in $\mathbb{R}^2\setminus\{(-1,0)\}$

Bordo di D: $\partial D=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$

1) $\Gamma_1=\{(x,-1): x\in[1,3]\}$,

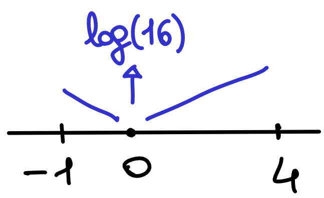
$h(x)=f(x,-1)=\log((x+1)^2+1)$



$$\boxed{f(1,-1)=\log(5),\ f(3,-1)=\log(17)}$$

2) $\Gamma_2=\{(3,y): y\in(-1,4)\}$,

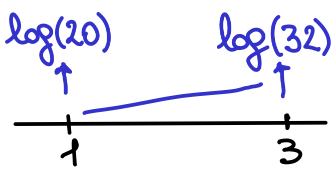
$h(y)=f(3,y)=\log(16+y^2)$



$$\boxed{f(3,0)=\log(16)}$$

3) $\Gamma_3=\{(x,4): x\in[1,3]\}$,

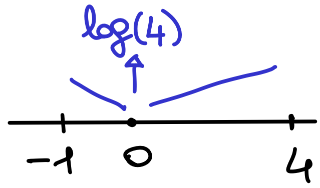
$h(x)=f(x,4)=\log((x+1)^2+16)$



$$\boxed{f(1,4)=\log(20),\ f(3,4)=\log(32)}$$

4) $\Gamma_4 = \{(1,y) : y \in (-1,4)\}$,

$$h(y) = f(1,y) = \log(4+y^2)$$



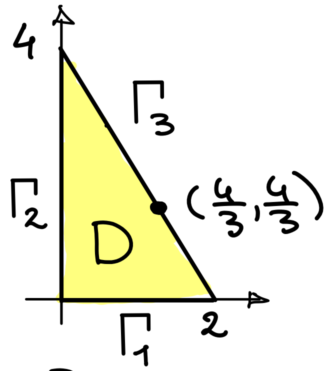
$$\boxed{f(1,0) = \log(4)}$$

Confrontando i valori trovati si ha che $(1,0)$ è un punto di minimo assoluto con valore $\log(4)$ e $(3,4)$ è un punto di massimo assoluto con valore $\log(32)$.

5.b

$f(x,y) = x^2 y$ Max/min in $D = \{(x,y) : 2x+y \leq 4, x \geq 0, y \geq 0\}$?

$f \in C^2(\mathbb{R}^2)$, D è compatto.



Punti stazionari interni a D:

$$\nabla f(x,y) = (2xy, x^2) = (0,0)$$

$x = 0$ e $\forall y \in \mathbb{R}$.

Nessun punto stazionario è interno a D.

Bordo di D: $\partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$

1) $\Gamma_1 = \{(x,0) : x \in [0,2]\}$

$h(x) = \boxed{f(x,0) = 0}$ costante

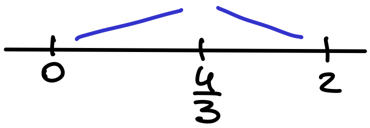
2) $\Gamma_2 = \{(0,y) : y \in [0,4]\}$

$h(x) = \boxed{f(0,y) = 0}$ costante

3) $\Gamma_3 = \{(x, 4-2x) : x \in (0,2)\}$

$$h(x) = f(x, 4-2x) = x^2 \cdot (4-2x) = 4x^2 - 2x^3$$

$h'(x) = 8x - 6x^2 = 2x(4-3x)$

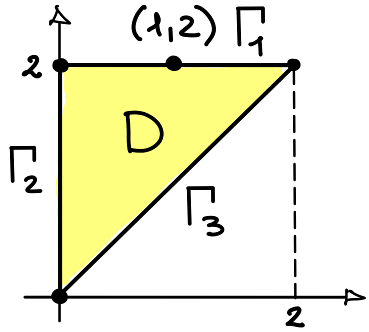


$$\boxed{f\left(\frac{4}{3}, \frac{4}{3}\right) = \left(\frac{4}{3}\right)^3 = \frac{64}{27}}$$

Confrontando i valori trovati si ha che i punti di $\Gamma_1 \cup \Gamma_2$ sono punti di minimo assoluto con valore 0 e $\left(\frac{4}{3}, \frac{4}{3}\right)$ è un punto di massimo assoluto con valore $\frac{64}{27}$.

$\boxed{5.C}$ $f(x,y) = x^2 + 2y^2 - xy$ Max/min in $D = \{(x,y) : 0 \leq x \leq y \leq 2\}$?

$f \in C^2(\mathbb{R}^2)$, D è compatto.



Punti stazionari interni a D:

$$\nabla f(x,y) = (2x - y, 4y - x) = (0,0)$$

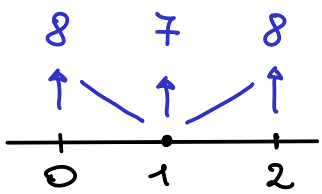
da cui si ottiene $(0,0)$ che non è interno a D.

Bordo di D: $\partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$

1) $\Gamma_1 = \{(x,2) : x \in [0,2]\}$

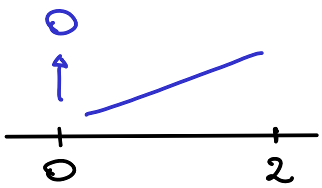
$h(x) = f(x,2) = x^2 + 8 - 2x$

$h'(x) = 2x - 2$



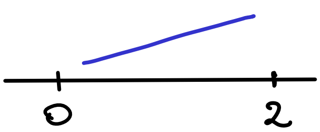
2) $\Gamma_2 = \{(0,y) : y \in [0,2)\}$

$h(y) = f(0,y) = 2y^2$



3) $\Gamma_3 = \{(x,x) : x \in (0,2)\}$

$h(x) = f(x,x) = 2x^2$

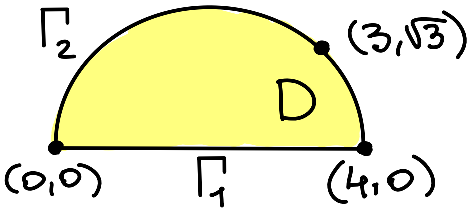


Confrontando i valori trovati si ha che $(2,2)$ e $(0,2)$ sono punti di massimo assoluto con valore 8 e $(0,0)$ è un punto di minimo assoluto con valore 0.

5.d $f(x,y)=x+\sqrt{3}y$ Max/min in $D=\{(x,y):x^2+y^2\leq 4x,\ y\geq 0\}$?

$\hookrightarrow (x-2)^2+y^2\leq 4$

$f\in C^2(\mathbb{R}^2)$, D è compatto.



$\partial D=\Gamma_1\cup\Gamma_2$

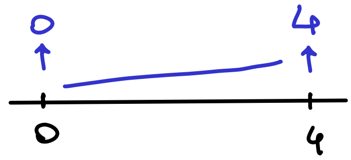
Non ci sono punti stazionari interni perché

$$\nabla f(x,y)=(1,\sqrt{3})\neq(0,0)$$

Bordo di D: $\partial D=\Gamma_1\cup\Gamma_2$ $\boxed{f(0,0)=0,\ f(4,0)=4}$

1) $\Gamma_1=\{(x,0):x\in[0,4]\}$

$h(x)=f(x,0)=x$



2) $\Gamma_2=\{(x,y):\ \underbrace{x^2+y^2-4x}_{=g(x,y)}=0,\ y>0\}$

$\nabla g(x,y)=(2x-4,2y)=(0,0)\Rightarrow(x,y)=(2,0)\notin\Gamma_2$ tutti regolari

Punti stazionari vincolati: $\begin{cases}\nabla f=\lambda\nabla g\\ g=0\end{cases}$

$$\begin{cases}1=\lambda(2x-4) & \rightarrow y=\sqrt{3}(x-2)\\ \sqrt{3}=\lambda(2y) & \rightarrow \lambda=\dfrac{\sqrt{3}}{2y}\\ x^2+y^2=4x & \rightarrow x^2+3(x-2)^2=4x,\ x^2+3x^2-12x+12=4x\end{cases}$$

$$x^2-4x+3=0 \rightarrow x=2\pm 1=\begin{cases}1\\3\end{cases}$$

$$\begin{cases} x=1 \\ y=-\sqrt{3} \\ \lambda=-\frac{1}{2} \end{cases} \cup \begin{cases} x=3 \\ y=\sqrt{3} \\ \lambda=\frac{1}{2} \end{cases} \qquad \begin{array}{l} (1,-\sqrt{3}) \notin \Gamma_2 \\ (3,\sqrt{3}) \in \Gamma_2 \end{array} \qquad \boxed{f(3,\sqrt{3})=6}$$

Confrontando i valori trovati si ha che $(0,0)$ è un punto di minimo assoluto con valore 0 e $(3,\sqrt{3})$ è un punto di massimo assoluto con valore 6.

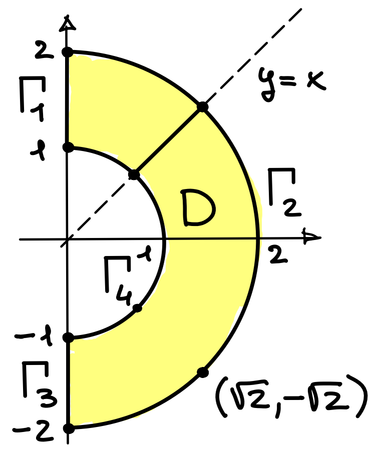
5.e $\quad f(x,y)=(y-x)^2 \quad$ Max/min in $D=\{(x,y): 1 \le x^2+y^2 \le 4, x \ge 0\}$?

$f \in C^2(\mathbb{R}^2)$, D è compatto.

Punti stazionari interni a D:

$$\nabla f(x,y)=(2x-2y,\, 2y-2x)=(0,0)$$

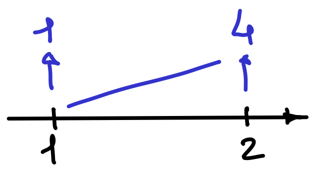
da cui $y=x$. Quindi i punti stazionari interni a D sono (x,x): $x \in \left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ e $\boxed{f(x,x)=0}$. Dato che $f \ge 0$ in $\mathbb{R}^2$ tali punti sono di minimo assoluto.



Bordo di D: $\partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$

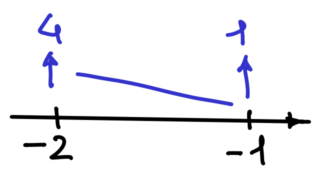
1) $\Gamma_1 = \{(0,y): y \in [1,2]\} \qquad f(0,y)=y^2$

con $\boxed{f(0,1)=1}$ e $\boxed{f(0,2)=4}$.



$\Gamma_3 = \{(0,y): y \in [-2,-1]\} \qquad f(0,y)=y^2$

con $\boxed{f(0,-2)=4}$ e $\boxed{f(0,-1)=1}$.



2) $\Gamma_2=\{(x,y): x^2+y^2=4, x>0\}$, $\Gamma_4=\{(x,y): x^2+y^2=1, x>0\}$

Sia $g(x,y)=x^2+y^2-r^2=0$ con $r=1,2$.

$\nabla g(x,y)=(2x,2y)\neq(0,0)$ per $(x,y)\in\Gamma_2\cup\Gamma_4$ tutti regolari

Punti stazionari vincolati:

$$\begin{cases}\cancel{2}x-\cancel{2}y=\lambda\cancel{2}x & \cdot y \to xy-y^2=\lambda xy\\ \cancel{2}y-\cancel{2}x=\lambda\cancel{2}y & \cdot x \to xy-x^2=\lambda xy\\ x^2+y^2=r^2 & \downarrow \text{sottraendo } x^2=y^2\end{cases}$$

$$\begin{cases}x=\pm y & x>0\\ 2x^2=r^2\end{cases} \Longrightarrow \left(\frac{r}{\sqrt{2}},\frac{r}{\sqrt{2}}\right),\ \left(\frac{r}{\sqrt{2}},-\frac{r}{\sqrt{2}}\right)$$

Per $r=1,2$ si ha che

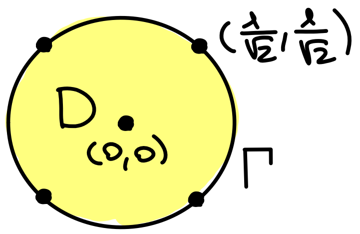
$$\boxed{\begin{array}{l} f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=f(\sqrt{2},\sqrt{2})=0\\ f\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)=2,\ f(\sqrt{2},-\sqrt{2})=8\end{array}}$$

Confrontando i valori ottenuti si ha che:

$(x,x): x\in\left[\frac{1}{\sqrt{2}},\sqrt{2}\right]$ sono punti di minimo assoluto con valore 0 e $(\sqrt{2},-\sqrt{2})$ è un punto di massimo assoluto con valore 8.

$\boxed{5.f}$ $f(x,y)=\dfrac{xy}{1+x^2+y^2}$ Max/min in $D=\{(x,y): x^2+y^2\leq 1\}$?

$f\in C^2(\mathbb{R}^2)$, D è compatto.



$$\partial D=\Gamma=\{(x,y): x^2+y^2=1\}$$
$$=\{(\cos\theta,\sin\theta): \theta\in[0,2\pi)\}$$

Punti stazionari interni a D:

$$\nabla f(x,y)=\left(\frac{y(1+x^2+y^2)-2x^2y}{(1+x^2+y^2)^2},\frac{x(1+x^2+y^2)-2xy^2}{(1+x^2+y^2)^2}\right)=(0,0)$$

$$\begin{cases} y(1-x^2+y^2)=0 \\ x(1+x^2-y^2)=0 \end{cases} \to y=0\,,\ y^2=x^2-1$$

$$\begin{cases} y=0 \\ x(1+x^2)=0 \end{cases} \cup \begin{cases} y^2=x^2-1 \to y^2=-1 \text{ impossibile} \\ x\cdot 2=0 \to x=0 \end{cases}$$

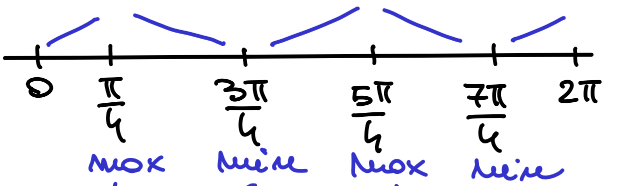
$\to x=0$

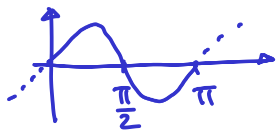
L'unico punto stazionario è $(0,0)$ ed è interno a D.

$$\boxed{f(0,0)=0}$$

Bordo di D: consideriamo per $\theta\in[0,2\pi)$

$$h(\theta)=f(\cos\theta,\operatorname{sen}\theta)=\frac{1}{2}\cos\theta\cdot\operatorname{sen}\theta=\frac{1}{4}\operatorname{sen}(2\theta)$$





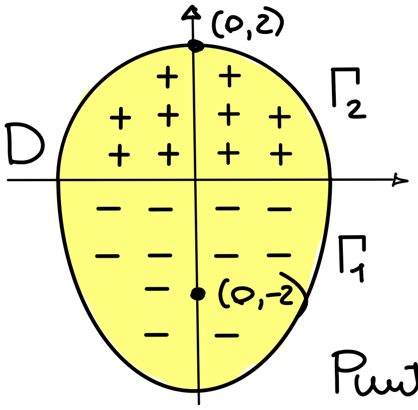
$$\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)\quad\left(-\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)\quad\left(-\frac{1}{\sqrt2},-\frac{1}{\sqrt2}\right)\quad\left(\frac{1}{\sqrt2},-\frac{1}{\sqrt2}\right)$$

$$\boxed{f\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)=f\left(-\frac{1}{\sqrt2},-\frac{1}{\sqrt2}\right)=\frac{1}{4}}\qquad\boxed{f\left(-\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)=f\left(\frac{1}{\sqrt2},-\frac{1}{\sqrt2}\right)=-\frac{1}{4}}$$

Confrontando i valori ottenuti si ha che:
$\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)$ e $\left(-\frac{1}{\sqrt2},-\frac{1}{\sqrt2}\right)$ sono punti di massimo assoluto con valore $\frac{1}{4}$ e $\left(-\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)$, $\left(\frac{1}{\sqrt2},-\frac{1}{\sqrt2}\right)$ sono punti di minimo assoluto con valore $-\frac{1}{4}$.

$\boxed{5.9}$ $f(x,y)=y(y-x^2+4)$ Max/min in
$D=\{(x,y): x^2-4\leq y\leq\sqrt{4-x^2}\}$?

$f\in C^2(\mathbb{R}^2)$, D è compatto.



Bordo di D: $\partial D=\Gamma_1\cup\Gamma_2$

$\Gamma_1=\{(x,x^2-4): x\in[-2,2]\}$

$\Gamma_2=\{(x,y): x^2+y^2=4, y\in(0,2]\}$

Punti stazionari interni a D:

$$\nabla f(x,y)=(-2xy,\ 2y-x^2+4)=(0,0)$$

$$\begin{cases}-2xy=0 \to x=0,\ y=0\\ 2y-x^2+4=0\end{cases}\iff\begin{cases}x=0\\ y=-2\end{cases}\ (0,-2)\qquad \begin{cases}y=0\\ x=\pm2\end{cases}\ (\pm2,0)$$

Solo $(0,-2)$ è interno a D e $\boxed{f(0,-2)=-4}$

Bordo di D:

1) In Γ_1 la funzione vale $\boxed{f(x,x^2-4)=0}$

2) Per Γ_2 usiamo i moltiplicatori con

$$g(x,y)=x^2+y^2-4=0.$$

$$\begin{cases}-\cancel{2}xy=\lambda\cdot\cancel{2}x \to x(\lambda+y)=0 \to x=0,\ \lambda=-y\\ 2y-x^2+4=\lambda\cdot2y\\ x^2+y^2=4\end{cases}$$

$$\iff\begin{cases}x=0\\ 2y+4=\lambda2y\\ y^2=4 \to y=\pm2\end{cases}\cup\begin{cases}\lambda=-y\\ 2y-x^2+4=-2y^2\\ x^2+y^2=4\end{cases}$$

$2y+y^2=-2y^2$

$y=0$ $\quad$ $y=-\frac{2}{3}$

$$\begin{cases} x=0 \\ y=2 \\ \lambda=2 \end{cases} \cup \begin{cases} x=0 \\ y=-2 \\ \lambda=0 \end{cases} \cup \begin{cases} y=0 \\ x=\pm 2 \\ \lambda=0 \end{cases} \cup \begin{cases} y=-\frac{2}{3} \\ x=\pm\frac{4\sqrt{2}}{3} \\ \lambda=\frac{2}{3} \end{cases}$$

$(0,2)\in\Gamma_2$ $\quad (0,-2)\notin\Gamma_2$ $\quad (\pm 2,0)\notin\Gamma_2$ $\quad \left(\pm\frac{4\sqrt{2}}{3},-\frac{2}{3}\right)\notin\Gamma_2$

$(0,2)$ è l'unico punto stazionario vincolato e

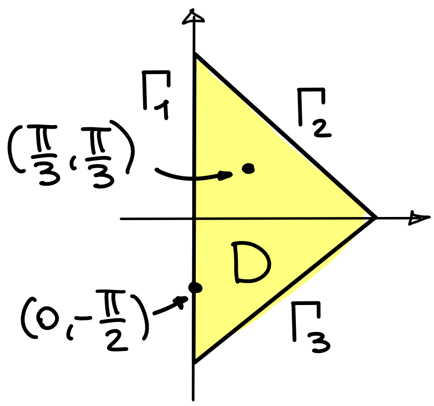
$$\boxed{f(0,2)=12}$$

Confrontando i valori, $(0,2)$ è un punto di massimo assoluto con valore 12 e $(0,-2)$ è un punto di minimo con valore -4.

$\boxed{5.h}$ $\quad f(x,y)=\operatorname{sen}(x)+\operatorname{sen}(y)+\operatorname{sen}(x+y)$

Max/min in $D=\{(x,y): x+|y|\le\pi,\ x\geq 0\}$?

$f\in C^2(\mathbb{R}^2)$, D è compatto.



Bordo di D: $\partial D=\Gamma_1\cup\Gamma_2\cup\Gamma_3$

$\Gamma_1=\{(0,y): y\in[-\pi,\pi]\}$

$\Gamma_2=\{(x,\pi-x): x\in(0,\pi]\}$

$\Gamma_3=\{(x,x-\pi): x\in(0,\pi)\}$

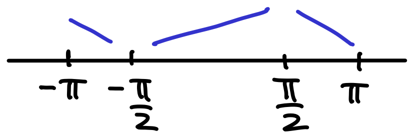
Punti stazionari: dall'esercizio 4.l del foglio 3, l'unico punto stazionario interno a D è

$\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ e $\boxed{f\left(\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{2}}$

Bordo di D:

1) $\Gamma_1=\{(0,y) : y\in[-\pi,\pi]\}$

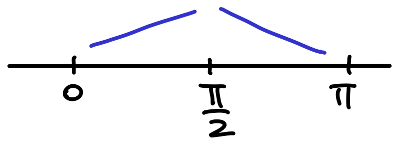
$h(y)=f(0,y)=2\,\mathrm{sen}(y)$



$$\boxed{f(0,\pi)=f(0,-\pi)=0\ ,\ f(0,-\tfrac{\pi}{2})=-2\ ,\ f(0,\tfrac{\pi}{2})=2}$$

2) $\Gamma_2=\{(x,\pi-x) : x\in(0,\pi]\}$

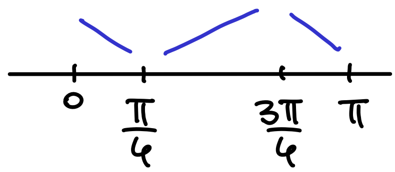
$h(x)=f(x,\pi-x)=2\,\mathrm{sen}(x)$



$$\boxed{f(\tfrac{\pi}{2},\tfrac{\pi}{2})=2\ ,\ f(\pi,\pi)=0}$$

3) $\Gamma_3=\{(x,x-\pi): x\in(0,\pi)\}$

$h(x)=f(x,x-\pi)=-\,\mathrm{sen}(2x)$



$$\boxed{f(\tfrac{\pi}{4},-\tfrac{3\pi}{4})=-1,\ f(\tfrac{3\pi}{4},-\tfrac{\pi}{4})=1}$$

Confrontando i valori, $(\frac{\pi}{3},\frac{\pi}{3})$ è un punto di massimo assoluto con valore $\frac{3\sqrt{3}}{2}$ e $(0,-\frac{\pi}{2})$ è un punto di minimo assoluto con valore -2.

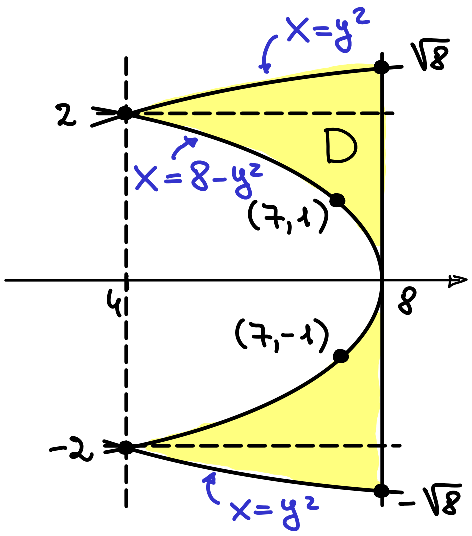
$\boxed{5.i}$ $f(x,y)=x^2+y^2-13x$

Max/min in $D=\{(x,y) : |x-2|\geq 2+|y^2-4|, 0<x\leq 8\}$?

$f\in C^2(\mathbb{R}^2)$, D è compatto. Se $0<x\leq 8$ allora

$$|x-2|\geq 2+|y^2-4| \begin{cases} \cancel{2}\underbrace{-x}_{<0}\geq \cancel{2}+\underbrace{|y^2-4|}_{\geq 0} & \text{se } 0<x<2 \quad \text{impossibile} \\ x-2\geq 2+|y^2-4| & \text{se } 2\leq x\leq 8 \end{cases}$$

$$x \geq 4 + |y^2-4| \begin{cases} x \geq \cancel{4} + y^2 - \cancel{4} & \text{se } |y| \geq 2 \\ x \geq 4+4-y^2 = 8-y^2 & \text{se } |y| \leq 2 \end{cases}$$



Bordo di D: $\partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$

$\Gamma_1 = \{(x,y) : g_1(x,y) = x + y^2 - 8 = 0, x \in [4,8]\}$

$\Gamma_2 = \{(x,y) : g_2(x,y) = y^2 - x = 0, x \in [4,8]\}$

$\Gamma_3 = \{(8,y) : y \in [-\sqrt{8}, \sqrt{8}]\}$

∂D è regolare a tratti.

$\nabla f(x,y) = (2x-13, 2y)$ è l'unico punto stazionario $\left(\frac{13}{2}, 0\right) \notin D$. Dato che D è compatto e f è continua i punti di max/min assoluto stanno in ∂D.

Γ_1) Moltiplicatori di Lagrange:

$$\begin{cases} 2x-13 = \lambda \cdot 1 \\ 2y = \lambda 2y \\ x+y^2 = 8 \end{cases} \to \lambda = 1 \vee y = 0 \qquad \begin{cases} \lambda = 1 \\ x = 7 \\ y = \pm 1 \end{cases} \cup \begin{cases} y = 0 \\ x = 8 \end{cases}$$

$4 \leq x \leq 8$

$$\boxed{f(7,\pm 1) = -41} \qquad \boxed{f(8,0) = -40}$$

Inoltre agli estremi $\boxed{f(4,\pm 2) = -32}$

Γ_2) Moltiplicatori di Lagrange:

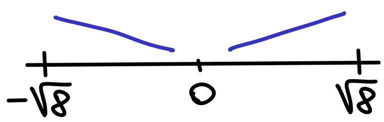
$$\begin{cases} 2x-13 = -\lambda \cdot 1 \\ 2y = \lambda 2y \\ y^2 = x \end{cases} \to \lambda = 1 \vee y = 0 \qquad \begin{cases} \lambda = 1 \\ x = 6 \\ y = \pm\sqrt{6} \end{cases} \cup \begin{cases} y = 0 \\ x = 0 \end{cases} \notin \Gamma_2$$

$4 \leq x \leq 8$

$$\boxed{f(6,\pm\sqrt{6}) = -36}$$

Inoltre agli estremi $\boxed{f(4,\pm 2) = -32, \; f(8,\pm\sqrt{8}) = -32}$

Γ_3) Restrizione $f(8,y) = -40 + y^2$



I punti $(8, \pm\sqrt{8})$ e $(8,0)$ sono già stati considerati.

Confrontando i valori, $(4, \pm 2)$ e $(8, \pm\sqrt{8})$ sono punti di massimo assoluto con valore -32 e $(7, \pm 1)$ sono punti di minimo assoluto con valore -41.