ANALISI MATEMATICA 2 - FOGLIO 2

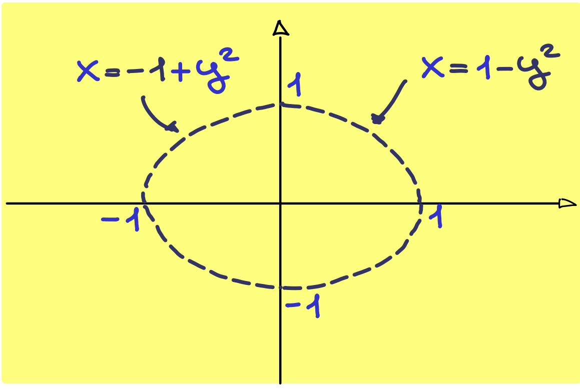
1.a $f(x,y) = \dfrac{e^{xy}}{|x|+y^2-1}$ Dominio D?

$$D = \{(x,y) \in \mathbb{R}^2 : |x|+y^2-1 \neq 0\}$$

Notiamo che

$$|x|+y^2-1=0 \iff |x| = 1-y^2 \geq 0 \iff \begin{cases} x = \pm(1-y^2) \\ \text{con } |y| \leq 1 \end{cases}$$

$|y|\leq 1$ parabole striscia orizzontale



1.b $f(x,y) = \dfrac{1}{\sqrt{2-|x|-|y|}}$ Dominio D?

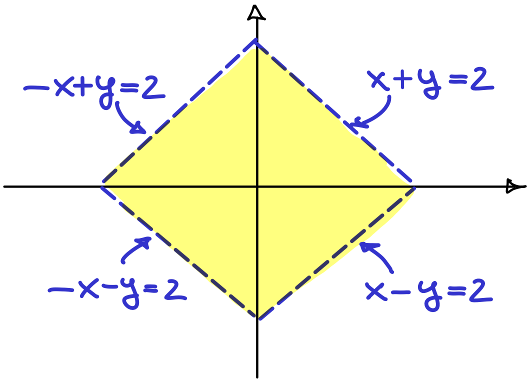
$$D = \{(x,y) \in \mathbb{R}^2 : |x|+|y| < 2\}$$

Se $x \geq 0$ e $y \geq 0$ $|x|+|y|=2 \iff x+y=2$ retta

Se $x \geq 0$ e $y \leq 0$ $|x|+|y|=2 \iff x-y=2$ retta

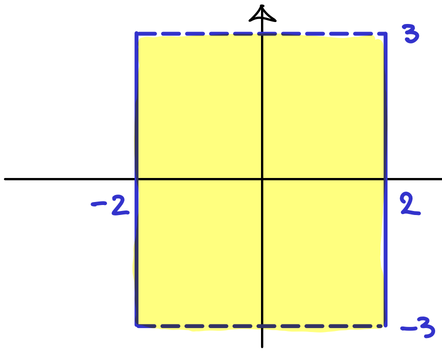
Se $x \leq 0$ e $y \geq 0$ $|x|+|y|=2 \iff -x+y=2$ retta

Se $x \leq 0$ e $y \leq 0$ $|x|+|y|=2 \iff -x-y=2$ retta



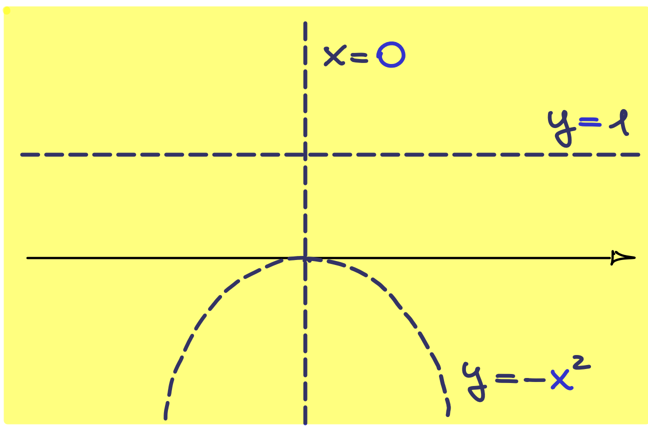
1.c $f(x,y)=\dfrac{\sqrt{4-x^2}}{\sqrt{9-y^2}}$ Dominio D?

$$D=\{(x,y)\in\mathbb{R}^2 : |x|\leq 2 \text{ e } |y|<3\}=[-2,2]\times(-3,3)$$



1.d $f(x,y)=\dfrac{\log|\underbrace{x^3+xy}_{x(x^2+y)}|}{y-1}$ Dominio D?

$$D=\{(x,y)\in\mathbb{R}^2 : x\neq 0 \text{ e } x^2+y\neq 0 \text{ e } y\neq 1\}$$



1.e $f(x,y)=\log\left(\operatorname{sen}\left(\pi\sqrt{x^2+y^2}\right)\right)$ Dominio D?

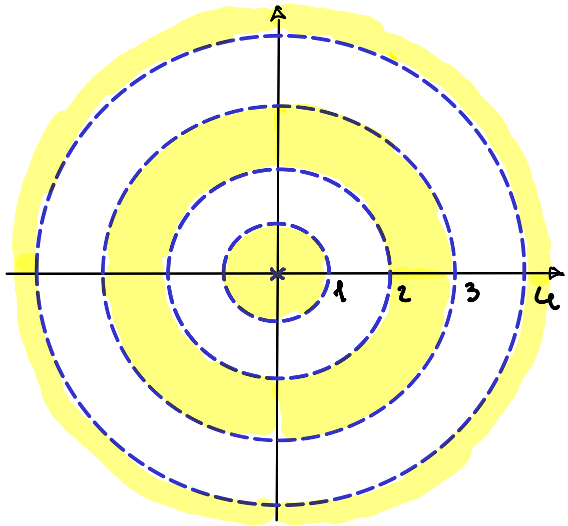
$$D=\{(x,y) : \operatorname{sen}(\pi\sqrt{x^2+y^2})>0\}$$

$$=\bigcup_{k=0}^{\infty}\{(x,y) : 2k<\sqrt{x^2+y^2}<2k+1\}$$

anello centrato in (0,0) di raggi 2k e 2k+1 senza bordo

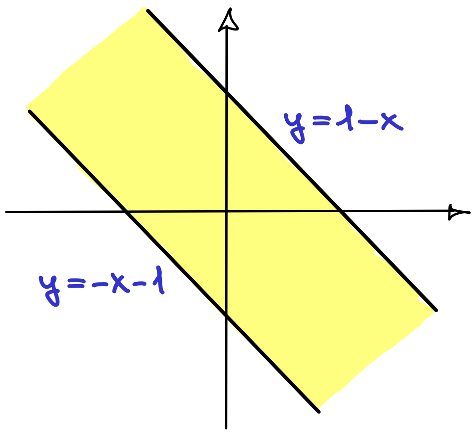
$\operatorname{sen}(t)>0$ per $t\in\bigcup_{k\in\mathbb{Z}}(2k\pi,(2k+1)\pi)$

$$\begin{array}{cccccccccccc} - & & + & & - & & + & & - & & + & - \\ & -2\pi & & -\pi & & 0 & & \pi & & 2\pi & & 3\pi \end{array}$$



$\boxed{1.f}$ $f(x,y) = \arcsin(x+y)$ Dominio D?

$$D = \{(x,y) : -1 \leq x+y \leq 1\}$$

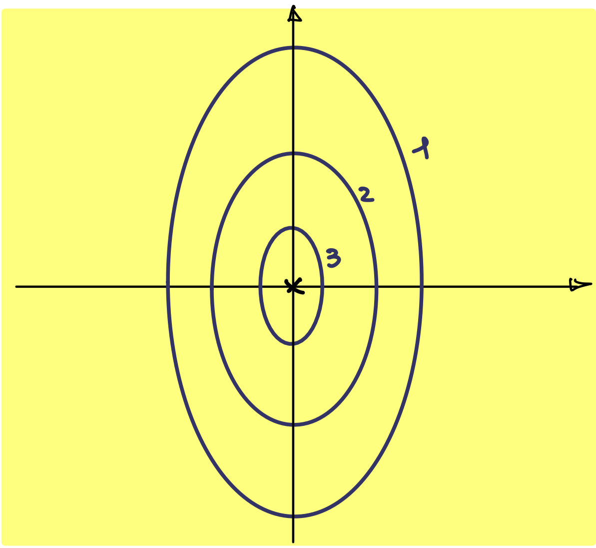


2.a $f(x,y)=\dfrac{1}{\sqrt{4x^2+y^2}}$ Curve di livello?

Per $(x,y)\in D=\mathbb{R}^2\setminus\{(0,0)\}$,

$$\frac{1}{\sqrt{4x^2+y^2}}=k>0 \iff \frac{1}{k^2}=4x^2+y^2 \iff \frac{x^2}{\left(\frac{1}{2k}\right)^2}+\frac{y^2}{\left(\frac{1}{k}\right)^2}=1$$

ellisse centrata in $(0,0)$ con semiassi $\frac{1}{2k}$ e $\frac{1}{k}$



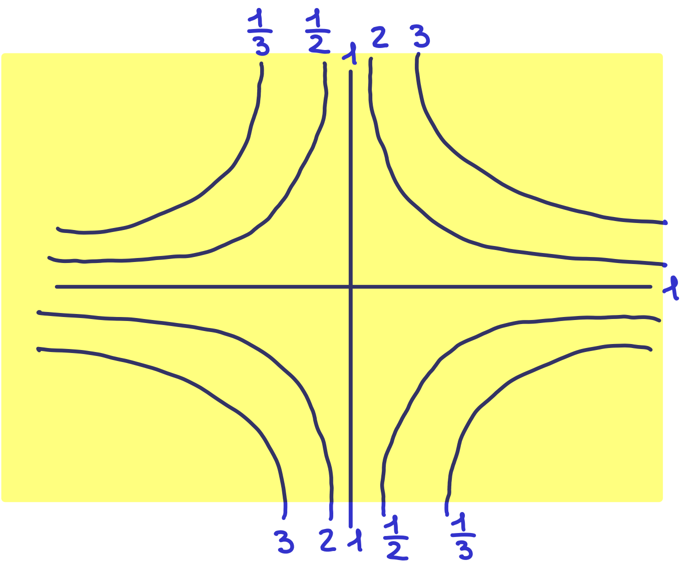
Al crescere di k le ellissi diventano sempre più piccole

2.b $f(x,y)=e^{xy}$ Curve di livello?

Per $(x,y)\in D=\mathbb{R}^2$

$$e^{xy}=k>0 \iff y=\log(k) \Rightarrow \begin{cases} x=0 \text{ oppure } y=0 \text{ se } k=1 \\ y=\dfrac{\log(k)}{x} \text{ altrimenti} \end{cases}$$

iperbole



Per $k=1$ c'è la coppia di rette $x=0$ e $y=0$. Le iperboli stanno nel 1° e 3° quadrante per $k>1$ e nel 2° e 4° quadrante per $0<k<1$.

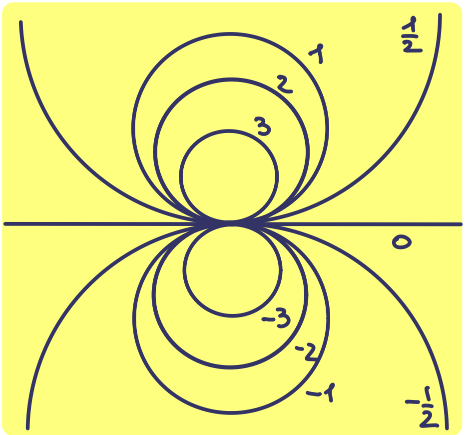
2.c $f(x,y)=\frac{4y}{x^2+y^2}$ Curve di livello?

Per $(x,y)\in D=\mathbb{R}^2\setminus\{(0,0)\}$,

$$\frac{4y}{x^2+y^2}=k \overset{k\neq 0}{\iff} x^2+y^2-\frac{4y}{k}=0 \iff x^2+\left(y-\frac{2}{k}\right)^2=\frac{4}{k^2}$$

$k=0 \Rightarrow y=0$

circonferenza di centro $(0,\frac{2}{k})$ e raggio $\frac{2}{k}$

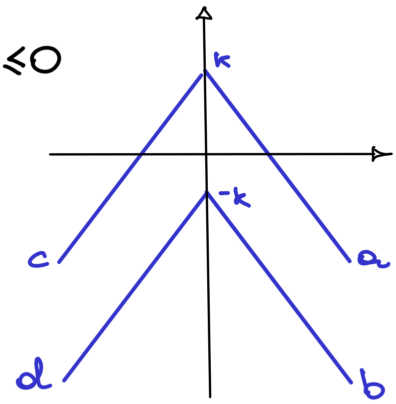


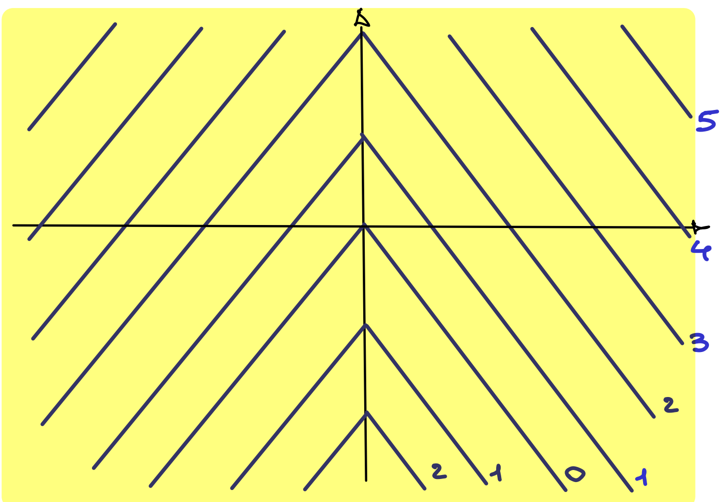
2.d $f(x,y)=||x|+y|$ Curve di livello?

Per $(x,y)\in D=\mathbb{R}^2$,

$$||x|+y|=k\geq 0 \iff \begin{cases} |x+y|=k & \text{se } x\geq 0 \\ |-x+y|=k & \text{se } x\leq 0 \end{cases}$$

$$\iff \begin{cases} x+y=k & \text{se } x\geq 0 & a) \\ x+y=-k & \text{se } x\geq 0 & b) \\ -x+y=k & \text{se } x\leq 0 & c) \\ -x+y=-k & \text{se } x\leq 0 & d) \end{cases}$$

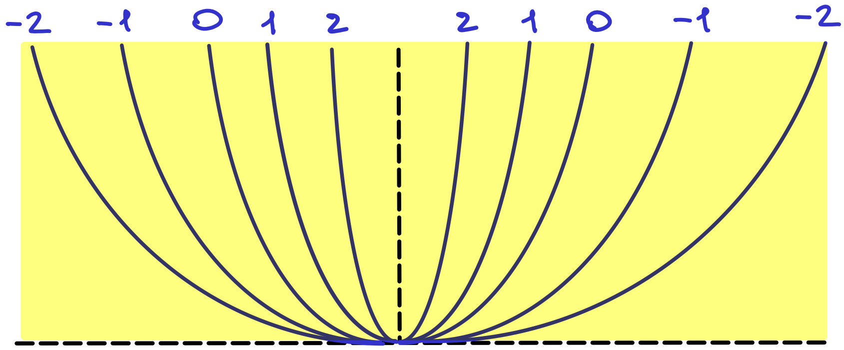




2.e $\quad f(x,y)=\log\left(\frac{y}{x^2}\right)$ Curve di livello?

Per $(x,y)\in D=\{(x,y): x\neq 0, y>0\}$

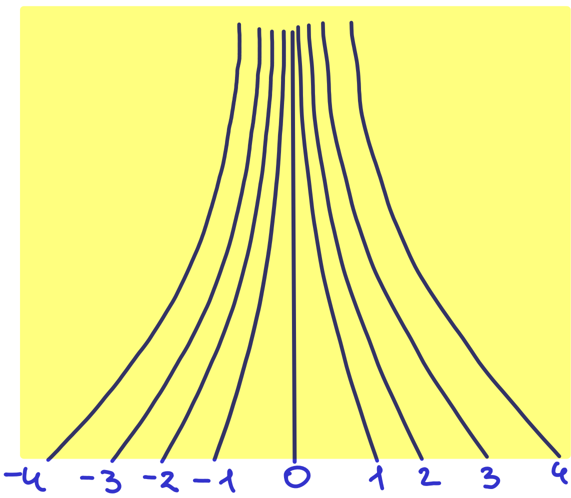
$$\log\left(\frac{y}{x^2}\right)=k \iff y=\underbrace{e^k}_{>0}\cdot x^2$$



2.f $\quad f(x,y)=(x+1)e^y$ Curve di livello?

Per $(x,y)\in D=\mathbb{R}^2$,

$$(x+1)e^y=k \iff x=-1+ke^{-y}$$



3.a $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{|x|+|y|} = 0$

Nel dominio $D=\mathbb{R}^2\setminus\{(0,0)\}$ se restringiamo f lungo le rette $x=0$ e $y=mx$ otteniamo sempre 0:

$$\underset{y\neq 0}{f(0,y)} = \frac{0}{|y|} = 0 \to 0 \quad \text{e} \quad f(x,mx) = \frac{m\cancel{x^2}^{\,|x|}}{\cancel{|x|}(1+|m|)} \to 0.$$

Quindi se il limite esiste vale 0.

Passando alle coordinate polari $x=\rho\cos\theta$ $y=\rho\,\mathrm{sen}\,\theta$

$$|f(x,y)| = \frac{\rho^2\,\overbrace{|\cos\theta\,\mathrm{sen}\,\theta|}^{\leq 1}}{\rho\,\underbrace{(|\cos\theta|+|\mathrm{sen}\,\theta|)}_{\geq m}} \leq \rho\cdot\frac{1}{m} \to 0 \quad \text{per } \rho\to 0$$

dove

$$m = \min_{\theta\in[0,2\pi]}\big(|\cos\theta|+|\mathrm{sen}\,\theta|\big) > 0$$

perché la funzione $\theta \to |\cos\theta|+|\mathrm{sen}\,\theta| > 0$ è continua nel compatto $[0,2\pi]$

In alternativa si può anche dire che per $(x,y)\to(0,0)$

$$0 \leq \frac{|xy|}{|x|+|y|} = |x|\cdot\underbrace{\left(\frac{|y|}{|x|+|y|}\right)}_{\leq 1} \leq |x| \to 0.$$

3.b $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x|(1+y)}{\sqrt{x^2+y^2}}$ = non esiste

In $D=\mathbb{R}^2\setminus\{(0,0)\}$, consideriamo le restrizioni lungo le rette passanti per $(0,0)$:

$$\underset{y\neq 0}{x=0}, \quad f(0,y) = \frac{0}{|y|} = 0 \to 0 \quad \text{per } (x,y)\to(0,0)$$

$$\underset{x\neq 0}{y=mx}, \quad f(x,mx) = \frac{\cancel{|x|}(1+mx)}{\cancel{|x|}\sqrt{1+m^2}} \to \frac{1}{\sqrt{1+m^2}} \neq 0$$

Dato che i due limiti sono diversi si conclude che il limite dato non esiste.

3.c

$$\lim_{(x,y)\to(0,0)} \frac{x\log(1+x^2+2y^2)}{3x^2+y^2}=0$$

In $D=\mathbb{R}^2\setminus\{(0,0)\}$, per $(x,y)\to(0,0)$ si ha che

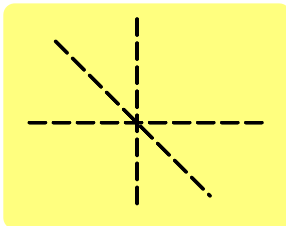
$$\frac{x\log(1+x^2+2y^2)}{3x^2+y^2}=\underbrace{\left(\frac{\log(1+x^2+2y^2)}{x^2+2y^2}\right)}_{\to 1}\cdot\underbrace{\left(\frac{x\cdot(x^2+2y^2)}{3x^2+y^2}\right)}_{\to 0}\to 0$$

perché $\lim\limits_{t\to0}\frac{\log(1+t)}{t}=1$ e

$$\left|\frac{x\cdot(x^2+2y^2)}{3x^2+y^2}\right|=|x|\cdot\underbrace{\left(\frac{x^2}{3x^2+y^2}\right)}_{\le 1}+2|x|\cdot\underbrace{\left(\frac{y^2}{3x^2+y^2}\right)}_{\le 1}\le 3|x|\to 0$$

3.d

$$\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)\sin\left(\frac{1}{xy^2}\right)}{e^{x+y}-1}=0$$



In $D=\{(x,y)\in\mathbb{R}^2:\ x\neq0 \text{ e } y\neq0 \text{ e } x+y\neq0\}$

per $(x,y)\to(0,0)$ si ha che

$$\frac{(x^2-y^2)\sin\left(\frac{1}{xy^2}\right)}{e^{x+y}-1}=\underbrace{\left(\frac{x+y}{e^{x+y}-1}\right)}_{\to 1}\cdot\frac{\cancel{(x+y)}\overbrace{(x-y)}^{\to 0}}{\cancel{x+y}}\cdot\underbrace{\left(\sin\left(\frac{1}{xy^2}\right)\right)}_{|\cdot|\le 1}\to 0$$

perché $\lim\limits_{t\to0}\frac{e^t-1}{t}=1$

3.e

$$\lim_{(x,y)\to(0,0)} \frac{x^2(1-y^2)+y^4}{x^2+y^4}=1$$

Per $(x,y)\to(0,0)$,

$$\frac{x^2(1-y^2)+y^4}{x^2+y^4}=1-\underbrace{\left(\frac{x^2y^2}{x^2+y^4}\right)}_{\to 0}\to 1$$

perché

$$0\le\frac{x^2y^2}{x^2+y^4}=\underbrace{\left(\frac{x^2}{x^2+y^4}\right)}_{\le 1}y^2\le y^2\to 0$$

3.f

$$\lim_{(x,y)\to(0,0)} \frac{x y^2 \log(|x|/|y|)}{\sqrt{x^2+y^2}} = 0$$

Per $(x,y)\to(0,0)$ in $D=\mathbb{R}^2 \setminus \{(x,y): x=0 \text{ o } y=0\}$

$$\frac{x y^2 \log(|x|/|y|)}{\sqrt{x^2+y^2}} = \underbrace{\left(\frac{y^2}{\sqrt{x^2+y^2}}\right)}_{\to 0} \underbrace{x\cdot\log(|x|)}_{\to 0} - \underbrace{\left(\frac{xy}{\sqrt{x^2+y^2}}\right)}_{\to 0} \underbrace{y\cdot\log(|y|)}_{\to 0}$$

Si ricorda che $\lim_{t\to 0} t^\alpha \log(t) = 0 \ \forall \alpha>0$. Inoltre

$$\alpha+\beta>1 \text{ e } \alpha,\beta\geq 0 \implies 0 \leq \frac{|x^\alpha y^\beta|}{\sqrt{x^2+y^2}} \leq \rho^{\overbrace{(\alpha+\beta-1)}^{>0}} \longrightarrow 0.$$

3.g

$$\lim_{(x,y)\to\infty} \frac{x^4+y^2-2x+3y}{2+\sqrt{x^2+y^2}} = +\infty$$

Notiamo che

$$x^4 \geq x^2-1, \quad \pm x \leq |x| \leq \sqrt{x^2+y^2} \ \text{ e } \ \pm y \leq |y| \leq \sqrt{x^2+y^2}$$

e quindi

$$x^4+y^2-2x+3y \geq x^2-1+y^2-2\sqrt{x^2+y^2}-3\sqrt{x^2+y^2}$$

Infine per $\rho=\sqrt{x^2+y^2}\longrightarrow+\infty$,

$$\frac{x^4+y^2-2x+3y}{2+\sqrt{x^2+y^2}} \geq \frac{\rho^2-5\rho-1}{2+\rho} \longrightarrow +\infty.$$

3.h

$$\lim_{(x,y)\to\infty} \frac{x^6+3y^3}{1+x^2} = \text{non esiste}$$

Restringendo $f(x,y)=\dfrac{x^6+3y^3}{1+x^2}$ lungo la retta $x=0$ si ha che

$$\lim_{y\to\pm\infty} f(0,y) = \lim_{y\to\pm\infty} 3y^3 = \pm\infty \quad \text{limiti diversi}$$

3.i $$\lim_{(x,y)\to(1,1)} \frac{x^2+xy-2y^2}{x^2-y^2} = \frac{3}{2}$$

Per $(x,y)\to(1,1)$ in $D=\mathbb{R}^2\setminus\{(x,y): |x|\neq|y|\}$

$$\frac{x^2+xy-2y^2}{x^2-y^2} = 1+\frac{xy-y^2}{x^2-y^2} = 1+\frac{y}{x+y} \longrightarrow 1+\frac{1}{2}=\frac{3}{2}.$$

3.j $$\lim_{(x,y)\to(2,1)} \frac{x^2-4y}{x-2y} = \text{non esiste}$$

Posto $u=x-2$ e $v=y-1$,

$$\lim_{(x,y)\to(2,1)} \frac{x^2-4y}{x-2y} = \lim_{(u,v)\to(0,0)} \frac{(u+2)^2-4(v+1)}{u+2-2(v+1)}$$
$$= \lim_{(u,v)\to(0,0)} \frac{u^2+4u-4v}{u-2v}$$

Restringendo f al fascio di rette $u=mv$ con $m\neq 2$

$$\frac{u^2+4u-4v}{u-2v} = \frac{m^2v^2+4mv-4v}{(m-2)v} \xrightarrow[v\to 0]{} \frac{4(m-1)}{m-2} \quad \text{limiti diversi}$$

Ad esempio per $m=0$ il limite è 2 mentre per $m=1$ il limite è 0.

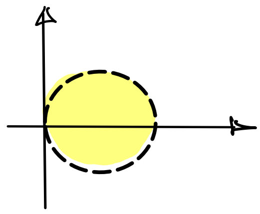
3.k $$\lim_{(x,y)\to(0,0)} \frac{x(1-\cos(y^2(x+2)))}{x^4+y^4} = 0$$

Dato che $t=y^2(x+2)\to 0$ e $\cos(t)=1-\frac{t^2}{2}+o(t^2)$

$$\frac{x(1-\cos(y^2(x+2)))}{x^4+y^4} = \frac{x\,(y^2(x+2))^2\left(\frac{1}{2}+o(1)\right)}{x^4+y^4}$$

$$= \frac{\rho\cos\theta\cdot\cancel{\rho^4}\sin^4\theta\cdot(4+o(1))\left(\frac{1}{2}+o(1)\right)}{\cancel{\rho^4}\,(\underbrace{\cos^4\theta+\sin^4\theta}_{\text{minimo positivo in }[0,2\pi]})} = \overset{\to 0}{\rho}\cdot\underbrace{(\ldots)}_{\text{limitata}} \to 0$$

3.l $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{2x-x^2-y^2}}$ = non esiste



$(x-1)^2+y^2<1$

Si noti che $D=\{(x,y): 2x-x^2-y^2>0\}$

Se restringiamo f lungo $y=0$ con $x>0$,

$$f(x,0)=0 \to 0 \quad \text{per } x\to 0^+$$

mentre se restringiamo f lungo $y=x^\alpha$ con $\alpha>0$ e $x>0$, si nota che per $\alpha=\frac{1}{2}$ e $x\to 0^+$,

$$f(x,x^\alpha)=\frac{x^\alpha}{\sqrt{2x-x^2-x^{2\alpha}}} \overset{\alpha=\frac{1}{2}}{=} \frac{x^{1/2}}{\sqrt{2x-x^2-x}} \sim \frac{\sqrt{x}}{\sqrt{x}} \to 1 \neq 0.$$

3.m $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3}$ = non esiste

Si noti che $D=\{(x,y): y\neq -x\}$.

Se restringiamo f lungo $y=0$ con $x>0$

$$f(x,0)=0 \to 0 \quad \text{per } x\to 0^+$$

mentre se restringiamo f lungo $y=-x+x^\alpha$ con $x>0$ e $\alpha>1$

$$f(x,-x+x^\alpha)=\frac{x^2(-x+x^\alpha)^2}{x^3+(-x+x^\alpha)^3}=\frac{x^4+o(x^4)}{\cancel{x^3}-\cancel{x^3}+3x^{2+\alpha}+o(x^{2+\alpha})}$$

$$=\frac{x^4+o(x^4)}{3x^{2+\alpha}+o(x^{2+\alpha})} \longrightarrow \begin{cases} 0 & \text{se } 1<\alpha<2 \\ 1/3 & \text{se } \alpha=2 \\ +\infty & \text{se } \alpha>2 \end{cases} \quad \text{per } x\to 0^+$$

Visto che per $\alpha\geq 2$ il limite è diverso da 0 il limite dato non esiste.

$\boxed{3.n}$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{|x|^3+|y|^3} = 0$$

Si noti che $D = \mathbb{R}^2 \setminus \{(0,0)\}$.

$$f(x,y) \overset{CP}{=} \frac{\rho^{4}\overbrace{(\cos\theta)^2}^{\leq 1}\overbrace{(\sin\theta)^2}^{\leq 1}}{\cancel{\rho^3}\underbrace{(|\cos\theta|^3+|\sin\theta|^3)}_{\text{minimo positivo in } [0,2\pi]}} = \rho\cdot\underbrace{(\cdots)}_{\text{limitata}} \longrightarrow 0 \text{ per } \rho\to 0.$$

$\boxed{3.o}$

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{\sin(x)+y} = \text{non esiste}$$

Si noti che $D = \{(x,y) : y \neq -\sin(x)\}$.

Per $x=0$ e $0<y<1$,

$$f(0,y) = \frac{y}{y} = 1 \longrightarrow 1 \quad \text{per } y\to 0^+.$$

Se invece $y = -x + x^\alpha$ con $\alpha>1$ per $x\to 0^+$,

$$f(x,-x+x^\alpha) = \frac{x^\alpha}{\cancel{x} - \frac{x^3}{6} + o(x^3) - \cancel{x} + x^\alpha}$$

$$\overset{\alpha=3}{=} \frac{x^3}{\frac{5}{6}x^3 + o(x^3)} \longrightarrow \frac{6}{5} \neq 1$$

e quindi il limite dato non esiste.

$\boxed{3.p}$

$$\lim_{(x,y)\to(0,0)} \frac{x^3(x+2y)}{x^4+y^4} = \text{non esiste}$$

Lungo la retta $y = mx$ per $x\to 0^+$ si ha che

$$f(x,mx) = \frac{\cancel{x^4}(1+2m)}{\cancel{x^4}(1+m^4)} \longrightarrow \frac{1+2m}{1+m^4} \begin{cases} \xrightarrow{m=0} 1 \\ \neq \\ \xrightarrow{m=1} \frac{3}{2} \end{cases}$$

il limite non esiste

$\boxed{3.q}$ $$\lim_{(x,y)\to(0,0)} \frac{\operatorname{sen}(x^2y) - x\operatorname{sen}(xy)}{x^4y^3} = \frac{1}{6}$$

Per $(x,y)\to(0,0)$, $\qquad \operatorname{sen}(t) = t - \frac{t^3}{6} + o(t^3)$ per $t\to 0$

$$\operatorname{sen}(x^2y) = x^2y - \frac{1}{6}x^6y^3 + o(x^6y^3)$$
$$x\operatorname{sen}(xy) = x\left(xy - \frac{1}{6}x^3y^3 + o(x^3y^3)\right)$$
$$= x^2y - \frac{1}{6}x^4y^3 + o(x^4y^3)$$

e quindi

$$\frac{\operatorname{sen}(x^2y) - x\operatorname{sen}(xy)}{x^4y^3} = \frac{\cancel{x^2y} - \frac{x^6y^3}{6} + o(x^6y^3) - \cancel{x^2y} + \frac{x^4y^3}{6} + o(x^4y^3)}{x^4y^3}$$
$$= -\frac{x^2}{6} + o(x^2) + \frac{1}{6} + o(1) \longrightarrow \frac{1}{6}.$$

$\boxed{3.r}$ $$\lim_{(x,y)\to(0,0)} \frac{x^2\log(1-y) + y\log(1+x^2)}{x^2y^2} = \text{non esiste}$$

Per $(x,y)\to(0,0)$, $\qquad \log(1+t) = t - \frac{t^2}{2} + o(t^2)$ per $t\to 0$

$$x^2\log(1-y) + y\log(1+x^2) = x^2\left(-y - \frac{(-y)^2}{2} + o(y^2)\right)$$
$$+ y\left(x^2 - \frac{x^4}{2} + o(x^4)\right)$$
$$= -\frac{x^2y^2}{2} + o(x^2y^2) - \frac{x^4y}{2} + o(x^4y)$$

non confrontabili

Si noti che se $y=x$, per $x\to 0^+$

$$f(x,x) = \frac{-\frac{x^4}{2} + o(x^4) - \frac{x^5}{2} + o(x^5)}{x^4} \longrightarrow -\frac{1}{2}$$

invece se $y=x^2$, per $x\to 0^+$

$$f(x,x^2) = \frac{-\frac{x^6}{2} + o(x^6) - \frac{x^6}{2} + o(x^6)}{x^4} \longrightarrow -1$$

$\neq$ Il limite non esiste