ANALISI MATEMATICA - LEZIONE 17

ALGEBRA DELL'O-PICCOLO

Per il simbolo dell'o-piccolo valgono le seguenti regole che derivano dalle proprietà dei limiti. Per $x \to 0$,

1) $\forall a \geq 0$ e $\forall c \neq 0$ $\quad c \cdot o(x^a) = o(c \cdot x^a) = o(x^a)$
2) $\forall b > a \geq 0$ e $\forall c$ $\quad c \cdot x^b + o(x^a) = o(x^a)$
3) $\forall b \geq a \geq 0$ $\quad o(x^b) + o(x^a) = o(x^a)$
4) $\forall a \geq -b$ $\quad x^b o(x^a) = o(x^{a+b})$
5) $\forall a, b \geq 0$ $\quad o(x^b) \cdot o(x^a) = o(x^{a+b})$
6) $\forall a, b \geq 0$ e $\forall c$ $\quad (c \cdot x^a + o(x^a))^b = c^b x^{ab} + o(x^{ab})$
7) $\forall a \geq 0$ e $\forall c$ $\quad o(c \cdot x^a + o(x^a)) = o(x^a)$

Per $x \to x_0$ basta sostituire $o(x^a)$ con $o((x-x_0)^a)$.

ESEMPI

- $$\lim_{x \to 0} \frac{\log(\cos(2x))}{e^x + e^{-x} - 2} = ?$$

Per $x \to 0$,

$\cos(t) = 1 - \frac{t^2}{2} + o(t^2)$

$e^t = 1 + t + \frac{t^2}{2} + o(t^2)$

$$= \frac{\log\left(1 - \frac{(2x)^2}{2} + o((2x)^2)\right)}{\left(\cancel{1} + \cancel{x} + \frac{x^2}{2} + o(x^2)\right) + \left(\cancel{1} + (\cancel{-x}) + \frac{(-x)^2}{2} + o((-x)^2)\right) - \cancel{2}}$$

$$= \frac{\log(1 - 2x^2 + o(x^2))}{x^2 + o(x^2)}$$

$\log(1+t) = t + o(t)$

$$= \frac{(-2x^2) + o(-2x^2)}{x^2 + o(x^2)} = \frac{-2x^2 + o(x^2)}{x^2 + o(x^2)} \longrightarrow -2.$$

- $$\lim_{x\to 0}\left(\frac{\operatorname{sen}(x)}{x}\right)^{1/x^2} = ?$$

Per $x\to 0$,

$$\left(\frac{\operatorname{sen}(x)}{x}\right)^{1/x^2} = \exp\left(\frac{1}{x^2}\log\left(\frac{\operatorname{sen}(x)}{x}\right)\right)$$

$\operatorname{sen}(x)=x-\frac{x^3}{6}+o(x^3)$

$$= \exp\left(\frac{1}{x^2}\log\left(1-\frac{x^2}{6}+o(x^2)\right)\right)$$

$\log(1+t)=t+o(t)$

$t=-\frac{x^2}{6}+o(x^2)\to 0$

$$= \exp\left(\frac{1}{x^2}\left(-\frac{x^2}{6}+\underbrace{o(x^2)+o\left(-\frac{x^2}{6}+o(x^2)\right)}_{=o(x^2)}\right)\right)$$

$$= \exp\left(-\frac{1}{6}+o(1)\right) \longrightarrow e^{-\frac{1}{6}}.$$

- $$\lim_{x\to 0}\frac{\operatorname{sen}(x^2)-\operatorname{sen}^2(x)}{x^2(\cos(x^2)-\cos^2(x))} = ?$$

Per $x\to 0$,

$$\frac{\operatorname{sen}(x^2)-\operatorname{sen}^2(x)}{x^2(\cos(x^2)-\cos^2(x))}$$

$\operatorname{sen}(t)=t-\frac{t^3}{6}+o(t^3)$

$\cos(t)=1-\frac{t^2}{2}+o(t^2)$

$$= \frac{x^2-\frac{(x^2)^3}{6}+o(x^6)-\left(x-\frac{x^3}{6}+o(x^3)\right)^2}{x^2\left(1-\frac{(x^2)^2}{2}+o(x^4)-\left(1-\frac{x^2}{2}+o(x^2)\right)^2\right)}$$

$$= \frac{x^2-\frac{x^6}{6}+o(x^6)-\left(x^2-2\frac{x^4}{6}+o(x^4)\right)}{x^2\left(1-\frac{x^4}{2}+o(x^4)-\left(1-2\frac{x^2}{2}+o(x^2)\right)\right)}$$

$$= \frac{\cancel{x^2}-\frac{x^6}{6}+o(x^6)-\left(\cancel{x^2}-2\frac{x^4}{6}+o(x^4)\right)}{x^2\left(\cancel{1}-\frac{x^4}{2}+o(x^4)-\left(\cancel{1}-2\frac{x^2}{2}+o(x^2)\right)\right)}$$

$$= \frac{\frac{1}{3}x^4+o(x^4)}{x^2\left(x^2+o(x^2)\right)} = \frac{\frac{1}{3}x^4+o(x^4)}{x^4+o(x^4)} \longrightarrow \frac{1}{3}.$$

- $$\lim_{x\to 1} \frac{\sqrt{x+3}-\sqrt[3]{3x+5}}{(\log(x))^2} = ?$$

Per $x \to 1$, $t = x-1 \to 0$ e

$$\frac{\sqrt{x+3}-\sqrt[3]{3x+5}}{(\log(x))^2} = \frac{\sqrt{4+t}-\sqrt[3]{8+3t}}{(\log(1+t))^2}$$

$\log(1+t) = t + o(t)$

$$= 2\,\frac{\left(1+\frac{t}{4}\right)^{1/2} - \left(1+\frac{3t}{8}\right)^{1/3}}{t^2+o(t^2)}$$

$(1+s)^{1/2} = 1 + \frac{s}{2} - \frac{s^2}{8} + o(s^2)$, $(1+s)^{1/3} = 1 + \frac{s}{3} - \frac{s^2}{9} + o(s^2)$

$$= \frac{2}{t^2+o(t^2)}\left(\cancel{1} + \cancel{\frac{t}{8}} - \frac{t^2}{128} + o(t^2) - \cancel{1} - \cancel{\frac{t}{8}} + \frac{t^2}{64} + o(t^2)\right)$$

$$= \frac{\frac{t^2}{64} + o(t^2)}{t^2 + o(t^2)} \longrightarrow \frac{1}{64}.$$

- $$\lim_{x\to +\infty} x\left(\left(1+\frac{1}{x}\right)^x - e\right) = ?$$

Per $x \to +\infty$, $t = \frac{1}{x} \to 0^+$ e

$$x\left(\left(1+\frac{1}{x}\right)^x - e\right) = \frac{(1+t)^{1/t} - e}{t} = \frac{1}{t}\left(\exp\left(\frac{\log(1+t)}{t}\right) - e\right)$$

$\log(1+t) = t - \frac{t^2}{2} + o(t^2)$

$$= \frac{1}{t}\left(\exp\left(1 - \frac{t}{2} + o(t)\right) - e\right)$$

$$= \frac{e}{t}\left(\exp\Big(\underbrace{-\frac{t}{2} + o(t)}_{=s\to 0}\Big) - 1\right)$$

$e^s = 1 + s + o(s)$

$$= \frac{e}{t}\left(\cancel{1} - \frac{t}{2} + o(t) - \cancel{1}\right)$$

$$= e\left(-\frac{1}{2} + o(1)\right) \longrightarrow -\frac{e}{2}.$$

Se avessimo usato l'espansione di ordine più basso $\log(1+t)=t+o(t)$ non si sarebbe potuto concludere:

$$x\left(\left(1+\frac{1}{x}\right)^{x}-e\right)=\frac{1}{t}\left(\exp\left(\frac{\log(1+t)}{t}\right)-e\right)$$

$\log(1+t)=t+o(t)$
$$=\frac{1}{t}\left(\exp(1+o(1))-e\right)$$
$$=\frac{e}{t}\left(\exp(\underbrace{o(1)}_{=s\to 0})-1\right)$$

$e^{s}=1+s+o(s)$
$$=\frac{e}{t}\left(\cancel{1}+o(1)-\cancel{1}\right)$$
$$=e\,\frac{o(1)}{t}\longrightarrow\ ?$$

Non è possibile confrontare gli infinitesimi $o(1)$ e t

- $$\lim_{n\to\infty} n^2\left(\exp\left(\frac{1}{n}-\frac{1}{n^2}\right)-\frac{\sqrt{n^2+2}}{n-1}\right)=\ ?$$

Per $n\to\infty$,

$$\exp\left(\frac{1}{n}-\frac{1}{n^2}\right)=1+\left(\frac{1}{n}-\frac{1}{n^2}\right)+\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n^2}\right)^2+o\left(\frac{1}{n^2}\right)$$

$e^{x}=1+x+\frac{x^2}{2}+o(x^2)$

$$=1+\frac{1}{n}-\frac{1}{n^2}+\frac{1}{2}\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)$$
$$=1+\frac{1}{n}-\frac{1}{2}\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)$$

Inoltre

$$\frac{\sqrt{n^2+2}}{n-1}=\cancel{n}\left(1+\frac{2}{n^2}\right)^{1/2}\cdot\frac{1}{\cancel{n}}\left(1-\frac{1}{n}\right)^{-1}$$

$(1+x)^{1/2}=1+\frac{x}{2}+o(x)$ $\qquad$ $(1-x)^{-1}=1+x+x^2+o(x^2)$

$$=\left(1+\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{n^2}+o\left(\frac{1}{n^2}\right)\right)\cdot\left(1+\frac{1}{n}+\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)\right)$$

$$= 1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$$
$$= 1 + \frac{1}{n} + \frac{2}{n^2} + o\left(\frac{1}{n^2}\right).$$

Quindi

$$n^2\left(\exp\left(\frac{1}{n} - \frac{1}{n^2}\right) - \frac{\sqrt{n^2+2}}{n-1}\right)$$
$$= n^2\left(\cancel{1} + \cancel{\frac{1}{n}} - \frac{1}{2}\cdot\frac{1}{n^2} + o\left(\frac{1}{n^2}\right) - \left(\cancel{1} + \cancel{\frac{1}{n}} + \frac{2}{n^2} + o\left(\frac{1}{n^2}\right)\right)\right)$$
$$= n^2\left(-\frac{1}{2}\cdot\frac{1}{n^2} - \frac{2}{n^2} + o\left(\frac{1}{n^2}\right)\right)$$
$$= -\frac{5}{2} + o(1) \longrightarrow -\frac{5}{2}.$$