Analisi Matematica 2 - Ing. Meccanica e Energetica
Soluzione della prova scritta del 18-9-2025

Esercizio 1. Sia $\displaystyle\sum_{k=1}^{\infty} \frac{\big(\log(5-x)\big)^k - 3k}{(k+1)!}$.

(a) Per quali $x \in \mathbb{R}$ la serie converge?

(b) Calcolare la somma della serie per $x = 2$.

(a) Ricordiamo che $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{k!}$ converge $\forall z \in \mathbb{R}$ a e^z.

Così, per $z = \log(5-x)$ con $x<5$, la serie converge a

$$\sum_{k=1}^{\infty}\left(\frac{z^k}{(k+1)!} - 3\frac{k}{(k+1)!}\right) \overset{j=k+1}{=} \sum_{j=2}^{\infty}\left(\frac{z^{j-1}}{j!} - 3\frac{j-1}{j!}\right)$$

telescopica oppure $=(e-1)-(e-1-1)$

$$= \frac{1}{z}\sum_{j=2}^{\infty}\frac{z^j}{j!} - 3\sum_{j=2}^{\infty}\left(\frac{1}{(j-1)!} - \frac{1}{j!}\right)$$

$$= \frac{1}{z}\left(e^z - 1 - z\right) - 3\cdot 1 = \frac{e^z - 1}{z} - 4.$$

(b) Se $x=2$ allora $z = \log(3)$ e la somma vale

$$\frac{e^{\log(3)} - 1}{\log(3)} - 4 = \frac{2}{\log(3)} - 4.$$

Esercizio 2. Sia $C = \{(x,0,z) : (x-5)^2 + z^2 \le 4,\ z \ge 0\}$ e sia D is solido ottenuto ruotando C di 180° in senso antiorario attorno all'asse z.

(a) Disegnare la sezione data dall'intersezione di D con il piano $z = 1$.

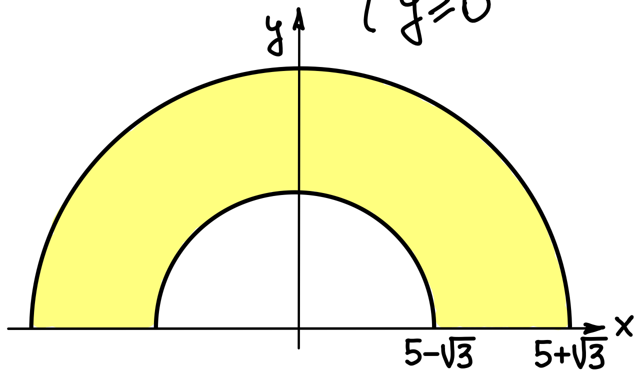
(b) Calcolare la coordinata z del baricentro di D.

(a) Abbiamo che

$$D = \left\{(x,y,z) : \left(\sqrt{x^2+y^2}-5\right)^2 + z^2 \le 4,\ y \ge 0,\ z \ge 0\right\}$$ un quarto di toro

e l'intersezione del piano $z=1$ con D è data da

$$\begin{cases} \left(\sqrt{x^2+y^2}-5\right)^2 + 1 \le 4 \\ y \ge 0 \end{cases} \iff \begin{cases} 5-\sqrt{3} \le \sqrt{x^2+y^2} \le 5+\sqrt{3} \\ y \ge 0 \end{cases}$$



(b) $$\iiint_D z\,dx\,dy\,dz \overset{CC}{=} \int_{\rho=5-2}^{5+2} \int_{\theta=0}^{\pi} \int_{z=0}^{\sqrt{4-(\rho-5)^2}} z\,dz\,d\theta\,\rho\,d\rho$$

$$= \frac{\pi}{2} \int_{5-2}^{5+2} \left(4-(\rho-5)^2\right)\rho\,d\rho \overset{t=\rho-5}{=} \frac{\pi}{2} \int_{-2}^{2} (4-t^2)(\underbrace{t}_{t\text{-disp}}+5)\,dt$$

$$= \frac{\pi}{\cancel{2}} \cancel{2} \cdot 5 \int_0^2 (4-t^2)\,dt = 5\pi \left[4t - \frac{t^3}{3}\right]_0^2 = \frac{80\pi}{3}.$$

Infine

$$\bar{z} = \frac{1}{|D|} \iiint_D z\,dx\,dy\,dz = \frac{8}{3\pi}$$

dove $|D| = \frac{1}{\cancel{4}} \pi \cdot \cancel{4} \cdot 2\pi \cdot 5 = 10\pi^2.$

Esercizio 3. Sia S la superficie parametrizzata da

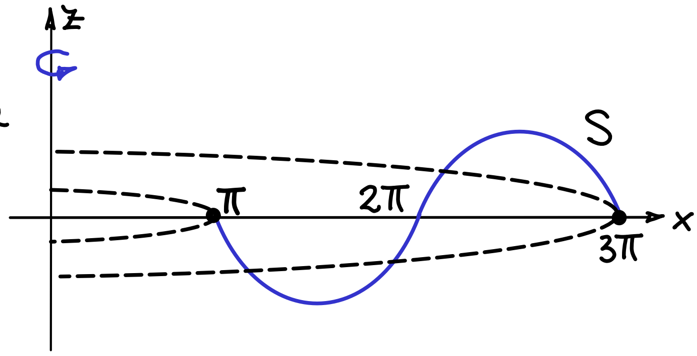
$$\boldsymbol{\sigma}(\rho,\theta) = (\rho\cos(\theta), \rho\sin(\theta), \sin(\rho))$$

con $(\rho,\theta) \in A = [\pi, 3\pi] \times [0, 2\pi]$.

(a) Calcolare $\displaystyle\int_\gamma \frac{|xy|}{4+z}\, ds$ dove $\gamma = \partial S$.

(b) Calcolare $\displaystyle\iint_S |z|\sqrt{\frac{2-z^2}{x^2+y^2}}\, dS$.

S è la rotazione completa attorno all'asse z del grafico di sen(x) con $x \in [\pi, 3\pi]$.



(a) $$\int_\gamma \frac{|xy|}{4+z}\,ds = \int_{C_\pi} \frac{|xy|}{4+z}\,ds + \int_{C_{3\pi}} \frac{|xy|}{4+z}\,ds = \frac{\pi^3}{2} + \frac{27\pi^3}{2} = 14\pi^3$$

dove $\gamma = \partial S = C_\pi \cup C_{3\pi}$ con

$$\vec{C}_R(t) = (R\cos t, R\,\mathrm{sen}\, t, 0)\ , t \in [0, 2\pi]$$

e

$$\int_{C_R} \frac{|xy|}{4+z}\,ds = \int_0^{2\pi} R^2 \frac{|\cos t\,\mathrm{sen}\, t|}{4+0}\, R\,dt = R^3 \int_0^{\pi/2} \cos t\,\mathrm{sen}\, t\,dt$$

$$= R^3 \left[\frac{1}{2}\mathrm{sen}^2\varphi\right]_0^{\pi/2} = \frac{R^3}{2}.$$

(b) Abbiamo che

$$\vec{\sigma}_\rho \times \vec{\sigma}_\theta = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \mathrm{sen}\,\theta & \cos\rho \\ -\rho\,\mathrm{sen}\,\theta & \rho\cos\theta & 0 \end{bmatrix}$$

$$= (\rho\cos\theta\cos\rho, \rho\,\mathrm{sen}\,\theta\cos\rho, \rho)$$

e $\|\vec{\sigma}_\rho \times \vec{\sigma}_\theta\| = \rho\sqrt{1+\cos^2\rho}.$

Così

$$\iint_S |z| \sqrt{\frac{2-z^2}{x^2+y^2}}\, dS = \int_{\rho=\pi}^{3\pi} \int_{\theta=0}^{2\pi} |\operatorname{sen}\rho| \frac{\sqrt{\overset{1+\cos^2\rho}{(2-\operatorname{sen}^2\rho)}}}{\cancel{\rho}} \cdot \cancel{\rho} \sqrt{1+\cos^2\rho}\, d\rho\, d\theta$$

$$= 2\pi \int_{\pi}^{3\pi} |\operatorname{sen}\rho| (1+\cos^2\rho)\, d\rho$$

$$= 2\pi \cdot 2 \int_{2\pi}^{3\pi} \operatorname{sen}\rho (1+\cos^2\rho)\, d\rho$$

$$= 4\pi \left[-\cos\rho - \frac{1}{3}\cos^3\rho \right]_{2\pi}^{3\pi} = 4\pi \cdot \frac{8}{3} = \frac{32\pi}{3}.$$

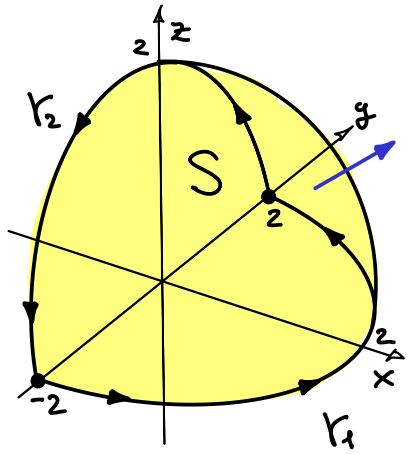
Esercizio 4. Sia $\boldsymbol{F}(x,y,z) = \left(xz, \dfrac{x}{x^2+y^2+1}, (y+3)z\right)$ e sia la superficie

$$S = \{(x,y,z) \,:\, x^2+y^2+z^2 = 4,\ x \geq 0,\ z \geq 0\}$$

orientata in modo che $\langle \boldsymbol{n}, \boldsymbol{k}\rangle \geq 0$ in ogni suo punto.

(a) Calcolare $\displaystyle\iint_S \langle \boldsymbol{F}, d\boldsymbol{S}\rangle$.

(b) Calcolare $\displaystyle\iint_S \langle \mathrm{rot}(\boldsymbol{F}), d\boldsymbol{S}\rangle$.



(a) $$\iint_S \langle \vec{F}, d\vec{S}\rangle = \iint_S \langle \vec{F}, \underbrace{\vec{n}}_{\frac{1}{2}(x,y,z)}\rangle dS$$

$$= \frac{1}{2}\iint_{S \,\leftarrow\, y\text{-simm}} \Big(x^2 z + \underbrace{\frac{xy}{x^2+y^2+1}}_{y\text{-disp}} + \underbrace{yz^2}_{y\text{-disp}} + 3z^2\Big) dS$$

$$\overset{CS}{=} \frac{1}{2}\int_{\varphi=0}^{\pi/2}\int_{\theta=-\pi/2}^{\pi/2} \left(8\,\mathrm{sen}^2\varphi\cos^2\theta\cos\varphi + 3\cdot 4\cos^2\varphi\right) 4\,\mathrm{sen}\,\varphi \, d\theta \, d\varphi$$

$$= \frac{\pi}{2}\cdot 16\int_0^{\pi/2} \mathrm{sen}^3\varphi\cos\varphi \, d\varphi + \pi\cdot 24\int_0^{\pi/2}\cos^2\varphi\,\mathrm{sen}\,\varphi\, d\varphi$$

$$= 8\pi\left[\frac{1}{4}\mathrm{sen}^4\varphi\right]_0^{\pi/2} + 24\pi\left[-\frac{1}{3}\cos^3\varphi\right]_0^{\pi/2} = 2\pi + 8\pi = 10\pi.$$

(b) Il bordo di S è costituito dalle semicirconferenze γ_1 e γ_2 con l'orientazione indicata in figura:

$$\vec{\gamma}_1(t) = (2\cos t, 2\,\mathrm{sen}\,t, 0) \quad \text{con } t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$\vec{\gamma}_2(t) = (0, 2\cos t, 2\,\mathrm{sen}\,t) \quad \text{con } t \in [0,\pi].$$

Per il teorema del rotore:

$$\iint_S \langle \mathrm{rot}(\vec{F}), d\vec{S}\rangle \overset{TR}{=} \int_{\gamma_1\cup\gamma_2} \langle \vec{F}, d\vec{s}\rangle = \frac{2\pi}{5} + \frac{16}{3}$$

dove

$$\int_{\gamma_1} \langle \vec{F}, d\vec{s} \rangle = \int_{-\pi/2}^{\pi/2} \frac{4\cos^2 t}{4+1}\, dt = \frac{4}{5} \cdot \frac{\pi}{2} = \frac{2\pi}{5}$$

e

$$\int_{\gamma_2} \langle \vec{F}, d\vec{s} \rangle = \int_{0}^{\pi} (2\cos t + 3)\, 2\,\mathrm{sen}\,t \cdot 2\cos t\, dt$$

$$= \left[-\frac{8}{3}\cos^3 t + \frac{12}{2}\,\mathrm{sen}^2 t \right]_0^{\pi} = \frac{16}{3} + 0 = \frac{16}{3}.$$